Eddy Viscosity for Time Reversing Waves in a Dissipative Environment

Josselin Garnier
Laboratoire de Statistique et Probabilités, Université Paul Sabatier, 118 Route de Narbonne, 31062 Toulouse Cedex 4, France

André Nachbin*
Instituto de Matemática Pura e Aplicada, Estrada D Castorina 110, Jardim Botânico, Rio de Janeiro, RJ 22460-320, Brazil
(Received 5 May 2004; published 4 October 2004)

We present new results for the time reversal of weakly nonlinear pulses traveling in a random dissipative environment. Also we describe a new theory for calculating the *eddy viscosity* for weakly nonlinear waves propagating over a random surface. The turbulent viscosity is calculated from first principles, namely, without imposing any stress-strain hypothesis. A viscous shallow water model is considered and its effective viscosity characterized. We also show that weakly nonlinear waves can still be time reversed under weak dissipation. Incoherently scattered signals are recompressed, both for time reversal in transmission as well as in reflection. Under the weakly nonlinear, weakly dissipative regime, dissipation only affects the refocused pulse profile regarding its amplitude, but its shape is not corrupted. Numerical experiments are presented.

DOI: 10.1103/PhysRevLett.93.154501 PACS numbers: 47.35.+i, 42.25.Dd, 47.27.Qb

Time-reversal refocusing for waves propagating in inhomogeneous media has been extensively studied in various contexts, as, for example, in ultrasound and underwater acoustics [1]. Applications include telecommunication [2] and imaging [3]. In a recent Letter [4] Fouque *et al.* considered the time reversal of one-dimensional waves in three new (related) regimes: (i) dispersive waves [5], (ii) weakly nonlinear waves, with the theory presented in detail in [6], and (iii) solitary waves, with detailed numerics provided in [7]. Dissipation is usually expected to be a serious limitation to time-reversal recompression efficiency [8]. In this Letter we consider one-dimensional weakly nonlinear waves in the unexplored regime of time reversal in a uniformly dissipative environment. It is shown for the first time the time reversal of weakly nonlinear waves in a viscous fluid. Motivated by this dissipative time-reversal study we describe the derivation of the *eddy (or turbulent) viscosity* [9] when propagating weakly nonlinear waves over a random (turbulent) surface. The novelty is that the *eddy viscosity* is characterized from first principles, for example, with no *a priori* hypothesis on stress-strain relationships [9]. We consider time reversal, both in transmission (TRT) and in reflection (TRR), for waves propagating in a dissipative model: the viscous shallow water equations $u_t + \alpha u u_x + \eta_x = \mu u_{xx}$, $\eta_t + [(1 + \varepsilon h + \alpha\eta)u]_x = 0$. In all cases the initial setup is given by a derivative of a Gaussian exactly as in [4]: $u(x, t)$ is the particle velocity and $\eta(x, t)$ is the wave elevation. This dimensionless system is obtained as a one-dimensional simplification of the Navier-Stokes equations under an incompressible flow with a hydrostatic pressure due to the long wave regime. Mass balance is built into the free surface condition for the evolution of the wave elevation $\eta(x, t)$ [10,11]. The characteristic wave amplitude is a_o and the typical depth is h_o. The nonlinearity parameter is $\alpha = a_o/h_o$ and the reference speed is $U = ga_o/c_o$ following the scaling in [11] where the linear shallow water speed is given by $c_o = (gh_o)^{1/2}$. The kinematic viscosity is denoted by μ. The parameter ε is the ratio of the standard deviation of the fluctuations of the depth over the average depth. It is assumed to be small. The fluctuations are modeled by the stationary zero-mean random process $h(x)$.

Let a pulse shaped wave propagate over a rapidly varying random region. Because of multiple scattering the transmitted coherent pulse will be followed by an incoherent coda. One will also observe a reflected incoherent signal. The following is shown: even in the presence of weak dissipation time reversing the corresponding data still leads to the recompression of its (high frequency) incoherent component. This recompression is not killed, nor corrupted, by viscous effects. Namely a striking feature is that the pulse shape is effectively preserved in the sense that the initial pulse profile can still be recovered through time reversal, hence not being corrupted along its Fourier content. For large values of the nonlinearity the initial profile shape can be only approximately recovered. This has been shown in [6] for nonlinear (inviscid) shallow water waves in the presence of a random topography and also for a dissipative Korteweg–de Vries (KdV) equation in a homogeneous medium [12]. In this Letter we describe the physics behind these results together with the main ingredients from the mathematical theory for the eddy viscosity, which will be published in full detail.

Time reversing in a homogeneous dissipative environment.—The time reversal of nonlinear acoustic waves in a homogeneous medium has been investigated experimentally by Tanter *et al.* [13]. They analyzed the nonlinear mechanism for energy transfer to higher harmonic components during forward propagation of a sinusoidal wave. The main goal of their experiments was to check for the reversibility of this energy transfer. The energy reversibility among harmonics is broken only for propagation longer than the shock formation distance. Both the acoustic and shallow water models are nonlinear conservation laws with a quadratic flux function and therefore analogous models [14]. In the special case of a polytropic gas with an adiabatic exponent equal to 2 the two models are identical. This issue was addressed in [4,6] for a broadband pulse in the presence of randomness. It was shown how the random medium regularizes the problem allowing for propagation beyond the shock distance. Just as in the super-resolution case [15], this is another instance where randomness helps in a dramatic fashion.

We repeat these homogeneous medium experiments, but now, in the presence of dissipation. In Dean and Dalrymple [16] (p. 262) normal ocean waves are in a regime where the Reynolds number is of the order $\mathrm{Re}^{-1} = O(10^{-7} - 10^{-8})$. By setting the dimensionless viscosity as $\mu = 10^{-4}$ we are considering stronger dissipation or equivalently dissipation over long propagation distances. Time-reversed water wave refocusing is a potential tool for waveform inversion, as, for example, in the cases of tsunamis [17]. Hence time reversal is most likely to be performed on a computer with a water wave model. This allows for considering two strategies for the time-reversed dynamics: (i) as usual, we perform it with the exact same model considered for the forward analysis; (ii) we switch off the dissipative mechanism, which can be easily done with a numerical model. By doing so we expect to have a better estimate of the initial profile's amplitude. This can be seen in Fig. 1 and its detailed version Fig. 2. In both cases we obtain the same energy reversibility among higher harmonics as was seen in the inviscid analysis presented in [6,13]. In Fig. 7 in [6] it was shown that the initial wave number range $k \in [-50, 50]$ generates high frequency harmonics in (effectively) $k \in [-250, 250]$. After time reversal the harmonics are back to the interval $[-50, 50]$. This means that the initial pulse *shape* (and therefore Fourier content) is effectively recovered, the only difference being a reduction in the amplitudes. This is due to dissipation, obviously acting over all wave numbers. When the viscosity is switched off, only for the time reversed experiment, the energy reversibility is more efficient. This can been seen in Fig. 1 and the detail presented in Fig. 2. Moreover, if nonlinearity is increased beyond a certain level energy reversibility through harmonics still takes place but the original wave number distribution is modified. This has been observed in [6] in the inviscid regime and also for a dissipative KdV equation [12].

Time reversing in a random dissipative environment.— The time reversal of the corresponding inviscid (weakly nonlinear) system was analyzed in [4,6]. It was shown that, to leading order, the transmitted nonlinear pulse is governed by a viscous Burgers equation. The "apparent viscosity" depends on statistical properties of the random medium, namely, the integral of the autocorrelation function of the fluctuations. This single parameter controls the level of coherent energy converted into incoherent energy for both the forward and back scattered signals. Hence stronger fluctuations increase the refocused amplitude in reflection and therefore decrease the refocused amplitude in transmission [6,18]. Therefore, in contrast with the kinematic viscosity, the *apparent viscosity* does not remove energy from the system: it converts coherent wave energy into incoherent fluctuations. In the linear regime this mechanism leads to pulse spreading and attenuation and is known as the O'Doherty-Anstey (ODA) approximation [5,11]. Linear Fourier analysis shows that, to leading order, the ODA "apparent attenuation" is diffusivelike and expressed through a Gaussian kernel. The transmitted wave is given by the convolution of the Gaussian kernel with the initial pulse's Fourier transform. Bearing in mind the discussion above we perform the time reversal in a random dissipative environment, by

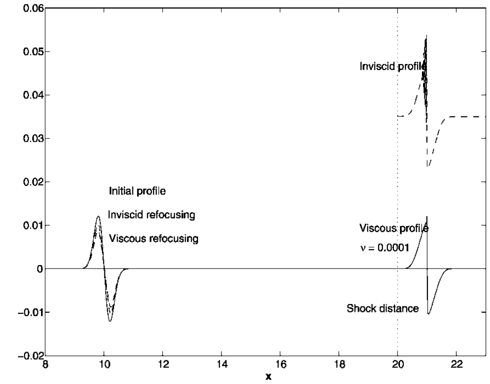

FIG. 1. Propagation in a homogeneous medium. To the right we have the transmitted pulse. To the left we have the initial profile together with refocusing in two regimes: in the presence of viscosity (smaller profile) and with viscosity switched off (intermediate profile). The upper right profile shows spurious numerical oscillations due to the shock. The regularizing effect, due to the small viscosity, is clearly seen in the time-reversed profile.

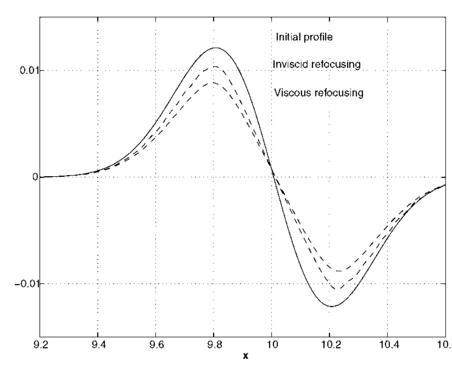

FIG. 2. A detail of Fig. 1.

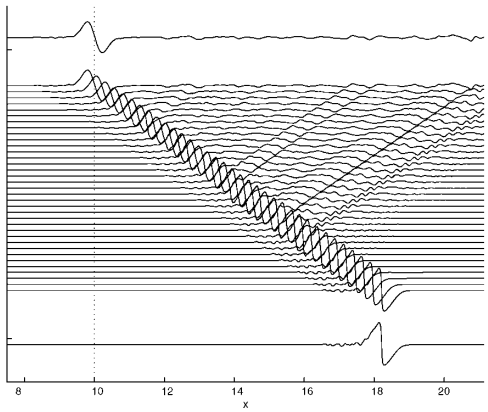

FIG. 3. Similar experiment to Fig. 1 but now in the presence of a (piecewise-linear) random medium located in [11,16.5]. Evolution is upward with the TRT profile as the bottom trace. The correlation length is $\ell = 0.1$ with $\varepsilon = 0.5$.

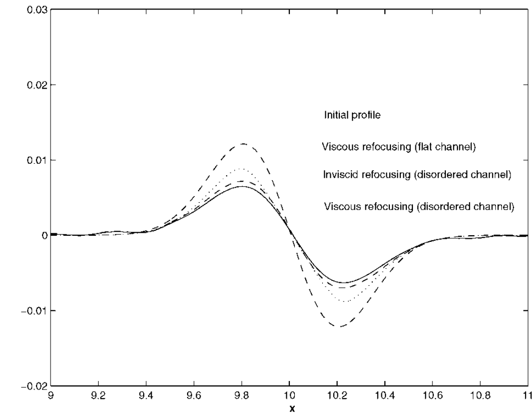

FIG. 4. A detail of Fig. 3 with additional TRT options.

considering the viscous shallow water system. In this case we have an *effective viscosity* [9] which arises from the combination of the kinematic viscosity together with the eddy viscosity. To leading order both components remove energy from the coherent wave front in a diffusivelike manner. Loosely speaking we can think of two Gaussian filters: one due to the physical viscosity (say G_μ) and the other due to the apparent (eddy) viscosity (say G_{μ_a}). The energy filtered by G_μ is lost forever and cannot be recovered. As discussed above, the energy filtered through G_{μ_a} can be recovered along the coherent wave front by time-reversal recompression. Even though these two filters are of a different nature, they are formally similar (i.e., both Gaussian). This leads to the striking fact that we can still recover the initial pulse *shape*, by time reversing waves in a dissipative environment. To leading order the dissipative mechanism does not destroy/interfere on the waveform inversion promoted by *incoherence recompression* and *energy reversibility* from higher harmonics. A TRT example is given in Fig. 3. We again explore the two strategies of time reversing *with* and *without* the fluid viscosity (c.f. Fig. 4). All forward experiments performed have viscous effects. We finally present a TRR experiment where, in the absence of the wave front, we see more clearly the recompression of the high frequency (incoherent) reflected wave. This signal feels more the effect of viscosity than the transmitted wave front. Nevertheless, the recompression process is not corrupted and the smooth initial profile emerges from this process. Note in Fig. 5 that as soon as refocusing takes place a traveling pulse emerges propagating to the left. See detail in Fig. 6.

Mathematical theory for the eddy viscosity.—The time-reversal theory for nonlinear hyperbolic waves makes use of the right and left Riemann invariants $A(x,t) = (\alpha u - 2c + 2)/\alpha$ and $B(x,t) = (\alpha u + 2c - 2)/\alpha$, corresponding to the underlying ($\varepsilon = 0$) nonlinear inviscid ($\mu = 0$) shallow water system [14]. The local propagation speed is $c = \sqrt{1 + \alpha\eta}$. The random fluid body is given by $H(x,t) = 1 + \varepsilon h(x) + \alpha\eta(x,t)$. Motivated by the time-reversal setup the shallow water system is rewritten in terms of these Riemann invariants

$$A_t + C^- A_x = \frac{\varepsilon[h(A+B)]_x}{2 + \alpha(B-A)/2} + \frac{\mu}{2}(A+B)_{xx},$$

$$B_t + C^+ B_x = -\frac{\varepsilon[h(A+B)]_x}{2 + \alpha(B-A)/2} + \frac{\mu}{2}(A+B)_{xx},$$

with $C^- \equiv -1 + \alpha(3A+B)/4$, $C^+ \equiv 1 + \alpha(A+3B)/4$. In the presence of stochastic forcing (i.e., $\varepsilon \neq 0$) they are not invariant along characteristics, being actually coupled due to the random scattering. We apply a convenient change of variables to rewrite the system in the framework moving with the right Riemann invariant, and in a new spatial variable related to the travel time. The system then reads as an upper triangular (two-by-two) system of partial differential equations. The equation for the leading order transmitted quantity is integrated in a Lagrangian fashion (with $\tau = t - x$) over the corresponding right characteristic. Assuming that $\varepsilon \ll 1$,

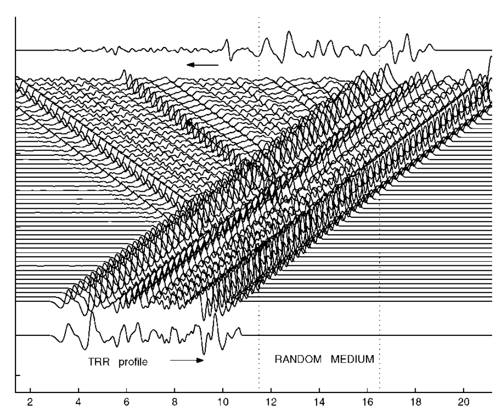

FIG. 5. Recompression of the incoherent reflected signal for a viscous fluid. The bottom trace is the time-reversed reflected signal. The top trace is the wave field at the proper refocusing time. The refocused pulse is a traveling wave propagating to the left.

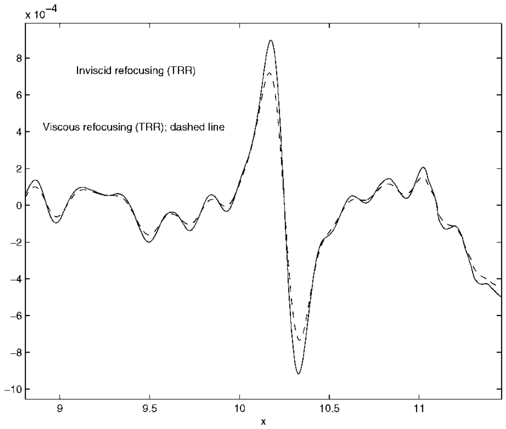

FIG. 6. A comparison of the refocused pulse from Fig. 5 (dashed line) with an experiment where viscosity was switched off during time reversion (solid line).

$\mu \sim \varepsilon^2$, and $\alpha \sim \varepsilon^2$, we can apply a limit theorem for differential equations with random coefficients [19]. This yields that the transmitted pulse front is governed by a Burgers-like equation $B_x = \mathcal{L}B + \alpha_e BB_\tau$, to leading order as $\varepsilon \downarrow 0$, where the effective nonlinearity is $\alpha_e = 3\alpha/4$. The linear pseudodifferential operator $\mathcal{L}$ can be written in the Fourier domain as the sum $\mathcal{L} = \mathcal{L}_r + \mathcal{L}_i$, with

$$\int \mathcal{L}_r B(\tau)e^{i\omega\tau}d\tau = -\frac{[b_r(2\omega) + 2\mu]\omega^2}{4}\int B(\tau)e^{i\omega\tau}d\tau,$$

$$\int \mathcal{L}_i B(\tau)e^{i\omega\tau}d\tau = -\frac{ib_i(2\omega)\omega^2}{4}\int B(\tau)e^{i\omega\tau}d\tau. \quad (1)$$

b_r and b_i are, respectively, the real and imaginary parts of b_0 defined as the Fourier transform of the positive lag part of the autocorrelation function of $\varepsilon h(x)$: $b_0(\omega) = \varepsilon^2 \int_0^\infty \langle h(0)h(x)\rangle e^{i\omega x}dx$. $\mathcal{L}_r$ results from the action of the kinematic viscosity and the apparent viscosity imposed by random forcing. The contribution $-b_r(2\omega)\omega^2/4$ is proportional to the power spectral density of the random process h that is nonnegative. It behaves like a pure diffusion for small frequencies, but it decays to zero for high frequencies. $\mathcal{L}_r$ can thus be seen as an effective *diffusion* operator and we have characterized the eddy viscosity from first principles. No *a priori* hypothesis was made [9]. $\mathcal{L}_i$ preserves the energy and it is an effective *dispersion* operator. It vanishes for small and high frequencies.

If the typical wavelength of the pulse is larger than the correlation radius of the medium, then b_0 can be considered as constant over the spectral range of the input pulse: $b_0(\omega) \equiv \mu_a$. As a result the early steps of the evolution are governed by the viscous Burgers equation $B_x = \mu_e B_{\tau\tau} + \alpha_e BB_\tau$ where the effective viscosity is $\mu_e = \mu/2 + \mu_a/4$. This system is known to support traveling pulses [14]. However, new frequencies generated by the nonlinear term may fall in the tail of the function b_0. Then the last equation may eventually fail describing the front pulse dynamics, and the equation with the pseudodifferential operator should be considered. In the present analysis we are left with only an equation for B because we considered right going initial data. In a more general scenario the asymptotics should lead to a coupled pair of effective Burgers-like equations (for A and B).

In conclusion, we have presented in this Letter the time-reversed refocusing of weakly nonlinear waves in an irreversible environment. The main results are the derivation (from first principles) of the eddy viscosity, together with experiments showing that dissipation does not destroy the recompression of the incoherent wave components. Harmonic energy reversibility is affected only in amplitude and therefore the pulse shape is not corrupted. This conclusion holds true for a spatially uniform dissipation, and extension to localized dissipation should reveal new features leading to original ways to image the dissipation distribution of a sample.

*Corresponding author.

[1] M. Fink, Sci. Am. **281**, No. 5, 67 (1999).
[2] A. Derode, A. Tourin, J. de Rosny, M. Tanter, S. Yon, and M. Fink, Phys. Rev. Lett. **90**, 014301 (2003).
[3] M. Fink, G. Montaldo, and M. Tanter, Ann. Rev. Biomed. Eng. **5**, 465 (2003).
[4] J.-P. Fouque, J. Garnier, J. Mũnoz Grajales, and A. Nachbin, Phys. Rev. Lett. **92**, 094502 (2004).
[5] J.-P. Fouque, J. Garnier, and A. Nachbin, SIAM J Appl. Math. (to be published).
[6] J.-P. Fouque, J. Garnier, and A. Nachbin, Physica D (Amsterdam) (to be published).
[7] J. Mũnoz Grajales and A. Nachbin, SIAM Multi. Model. Simul. (to be published).
[8] J.-F. Aubry, M. Tanter, J.-L. Thomas, M. Pernot, and M. Fink, J. Acoust. Soc. Am. **113**, 84 (2003).
[9] S. B. Pope, *Turbulent Flows* (Cambridge University Press, Cambridge, 2000).
[10] V. Casulli and R. Cheng, Int. J. Numer. Meth. Fluids **15**, 629 (1992).
[11] A. Nachbin and K. Sølna, Phys. Fluids **15**, 66 (2003).
[12] P. Milewski (personal communication).
[13] M. Tanter, J. Thomas, F. Coulouvrat, and M. Fink, Phys. Rev. E **64**, 016602 (2001).
[14] L. Debnath, *Nonlinear Partial Differential Equations for Scientists and Engineers* (Birkhäuser, Boston, 1997).
[15] P. Blomgren, G. Papanicolaou, and H. Zhao, J. Acoust. Soc. Am. **111**, 230 (2002).
[16] R. Dean and R. Dalrymple, *Water Wave Mechanics for Engineers and Scientists* (World Scientific, Singapore, 1984).
[17] C. Pires and P. Miranda, J. Geophys. Res. **106**, 19773 (2001).
[18] J.-F. Clouet and J.-P. Fouque, Wave Motion **25**, 361 (1997).
[19] R. Khasminskii, Theory Probab. Appl. **11**, 390 (1966).