# ON WAVES IN RANDOM MEDIA IN THE DIFFUSION-APPROXIMATION REGIME

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**Abstract.** The aim of this contribution is to present recent results obtained at the "Centre de Mathématiques Appliquées de l' Ecole Polytechnique" by the group working on waves in random media (F. Bailly, J. Chillan, J.F. Clouet, J.P. Fouque and J. Garnier). These results are based on various generalizations of classical diffusion-approximation results. In the first section we study the spreading of an acoustic pulse travelling through a randomly layered medium (Clouet-Fouque [8] and Chillan [6]). In the second section we present a justification of the parabolic and white noise approximation for waves in random media in the high frequency regime leading to a stochastic Schrödinger equation (Bailly-Clouet-Fouque [3] and Bailly [2]). The third section is devoted to the effect of a weak nonlinearity on a wave equation with a random potential (Garnier [12]). In the last section we study the amplification of an incoherent optical pulse propagating in a nonlinear Kerr medium [10].

**Key words.** acoustic waves, random media, diffusion-approximation, parabolic approximation, random Schrödinger equation, localization, nonlinear media.

**1. Spreading of an acoustic pulse travelling through a randomly layered medium.** We are interested in the following question: how the shape of a pulse has been modified when it emerges from a randomly layered medium? This analysis takes place in the general framework, based on separation of scales, introduced by G. Papanicolaou and his co-authors (see for instance [5] for the one-dimensional case or [1] for the three-dimensional case). We consider here the problem of acoustic propagation when the incident pulse wavelength is long compared to the correlation length of the random inhomogeneities but short compared to the size of the slab.

In this framework, it has already been proved in [1] (see also [7] for more details) that, when the random fluctuations are weak, the O'Doherty-Anstey theory is valid, i.e. the travelling pulse retains its shape up to a low spreading; futhermore, its shape is deterministic when observed from the point of view of an observer travelling at the same random speed as the wave while it is stochastic when the observer's speed is the mean speed of the wave.

We do not assume the fluctuations to be small but, as we are mainly concerned with the shape of the transmitted pulse, we suppose that the incident pulse has a constant amplitude but its energy is small. Our main result consists in a complete description of the asymptotic law of the emerging pulse: we prove a limit theorem which shows that the pulse spreads in a deterministic way (see also [15] for a result of the same nature).

In a first section we present the ideas of the proof in the one-dimensional

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case with no macroscopic variations of the medium and the noise only appearing in the density of the medium. In a second section we present the general result for one-dimensional media (obtained in [8] ) and in a third section we give the results obtained in the three-dimensional layered case (work in progress [6] ).

**1.1. The homogeneous one-dimensional case.** We consider an acoustic wave travelling in a one-dimensional random medium located in the region $0 \leq x \leq L$, satisfying the linear conservation laws:

$$\left\{ \begin{array}{ll} \rho(x)\frac{\partial u}{\partial t}(x,t) + \frac{\partial p}{\partial x}(x,t) & = 0 \\ \frac{1}{K(x)}\frac{\partial p}{\partial t}(x,t) + \frac{\partial u}{\partial x}(x,t) & = 0 \end{array} \right. \tag{1.1}$$

here $u(x,t)$ and $p(x,t)$ are respectively the speed and pressure of the wave, whereas $\rho(x)$ and $K(x)$ are the density and bulk modulus of the medium. In our simplified model we suppose that $K(x)$ is constant equal to 1 and that $\rho(x) = 1 + \eta(\frac{x}{\epsilon^2})$ where $\eta(\frac{x}{\epsilon^2})$ is the rapidly varying random coefficient describing the inhomogeneities. Since these coefficients are positive we suppose that $|\eta|$ is less than a constant strictly less than 1. Furthermore we assume that $\eta(x)$ is stationary, centered and mixing enough for $\frac{1}{\epsilon}\eta(\frac{x}{\epsilon^2})$ to be approximatively a white noise or, in other words, for $\int_0^y \frac{1}{\epsilon}\eta(\frac{x}{\epsilon^2})dx$ to converge in law to a brownian motion. In order to precise our boundary conditions we introduce the right and left going waves $A = u + p$ and $B = u - p$ which satisfy the following system of equations:

$$\frac{\partial}{\partial x}\left[\begin{array}{c} A \\ B \end{array}\right] = \left(\left[\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right] + \frac{1}{2}\eta(\frac{x}{\epsilon^2})\left[\begin{array}{cc} -1 & -1 \\ 1 & 1 \end{array}\right]\right)\frac{\partial}{\partial t}\left[\begin{array}{c} A \\ B \end{array}\right] \tag{1.2}$$

The slab of medium we are considering is located in the region $0 \leq x \leq L$ and at $t = 0$ an incident pulse is generated at the interface $x = 0$ between the random medium and the outside homogeneous medium. According to previous works ([5] or [1] ) we choose a pulse which is broad compared to the size of the random inhomogeneities but short compared to the macroscopic scale of the medium. There is no wave entering the medium at $x = L$.

$$\left\{ \begin{array}{ll} A(0,t) & = f(\frac{t}{\epsilon}) \\ B(L,t) & = 0 \end{array} \right. \tag{1.3}$$

where $f$ is a function with compact support and $C^\infty$ regularity. Note that the energy entering the medium $\epsilon \int f(t)^2 dt$ is of order $\epsilon$. We are interested in the transmitted pulse $A(L,t)$ around the arrival time $t = L$ and in the same scale as the entering pulse $A(0,t)$; therefore the quantity of interest is the windowed signal $A(L, L + \epsilon\sigma)_{\sigma \in (-\infty,\infty)}$ which will be given by the following centered and rescaled quantities:

$$\left\{\begin{array}{ll} a^\epsilon(x,\sigma) & = A(x, x+\epsilon\sigma) \\ b^\epsilon(x,\sigma) & = B(x, -x+\epsilon\sigma) \end{array}\right. \tag{1.4}$$

The solution of (1.2),(1.3) takes place in an infinite-dimensional space because of the variable $t$. So we perform the Fourier transforms:

$$\left\{\begin{array}{ll} \hat{a}^\epsilon(x,\omega) & = \int e^{i\omega\sigma} a^\epsilon(x,\sigma)d\sigma \\ \hat{b}^\epsilon(x,\omega) & = \int e^{i\omega\sigma} b^\epsilon(x,\sigma)d\sigma \end{array}\right.$$

In the frequency domain, with the change of variable (1.4), the equation (1.2) becomes:

$$\frac{d}{dx}\left[\begin{array}{c} \hat{a}^\epsilon \\ \hat{b}^\epsilon \end{array}\right] = \frac{i\omega}{2\epsilon}\eta(\frac{x}{\epsilon^2})\left[\begin{array}{cc} 1 & e^{-2i\omega\frac{x}{\epsilon}} \\ -e^{2i\omega\frac{x}{\epsilon}} & -1 \end{array}\right]\left[\begin{array}{c} \hat{a}^\epsilon \\ \hat{b}^\epsilon \end{array}\right] \tag{1.5}$$

with the boundary conditions $\hat{a}^\epsilon(0,\omega) = \hat{f}(\omega)$ and $\hat{b}^\epsilon(L,\omega) = 0$. The linearity of (1.5) enables us to replace these boundary conditions by:

$$\left\{\begin{array}{ll} \hat{a}^\epsilon(0,\omega) & = 1 \\ \hat{b}^\epsilon(L,\omega) & = 0 \end{array}\right. \tag{1.6}$$

and obtain the following representation for the transmitted pulse:

$$A(L, L+\epsilon\sigma) = a^\epsilon(L,\sigma) = \frac{1}{2\pi}\int e^{-i\omega\sigma}\hat{f}(\omega)\hat{a}^\epsilon(L,\omega)d\omega \tag{1.7}$$

where $(\hat{a}^\epsilon, \hat{b}^\epsilon)$ is now the solution of problem (1.5),(1.6).
The propagator matrix $Y^\epsilon(x,\omega)$ defined by:

$$\left[\begin{array}{c} \hat{a}^\epsilon(x,\omega) \\ \hat{b}^\epsilon(x,\omega) \end{array}\right] = Y^\epsilon(x,\omega)\left[\begin{array}{c} \hat{a}^\epsilon(0,\omega) \\ \hat{b}^\epsilon(0,\omega) \end{array}\right] \tag{1.8}$$

is the solution of equation (1.5) with the initial condition $Y^\epsilon(0,\omega) = Id_{\mathbb{C}^2}$.
If $(\alpha,\beta)$ is a solution of (1.5),(1.6) then $(\overline{\beta},\overline{\alpha})$ is another solution linearly independent from the previous one and one can write $Y^\epsilon(x,\omega)$ as:

$$Y^\epsilon(x,\omega) = \left[\begin{array}{cc} \alpha(x,\omega) & \overline{\beta}(x,\omega) \\ \beta(x,\omega) & \overline{\alpha}(x,\omega) \end{array}\right] \tag{1.9}$$

The trace of the matix appearing in the linear equation (1.5) being 0, we deduce that the determinant of $Y^\epsilon(x,\omega)$ is constant and equal to 1 which implies that $|\alpha(x,\omega)|^2 - |\beta(x,\omega)|^2 = 1$ for every $x$. Using (1.8) and (1.9) at $x = L$ and boundary conditions (1.6) we deduce that:

$$\left\{ \begin{array}{ll} \hat{a}^\epsilon(L,\omega) & = \frac{1}{\overline{\alpha}(L,\omega)} \\ \hat{b}^\epsilon(0,\omega) & = -\frac{\beta(L,\omega)}{\overline{\alpha}(L,\omega)} \end{array} \right. \tag{1.10}$$

In particular we have the following relation of conservation of energy:

$$|\hat{a}^\epsilon(L,\omega)|^2 + |\hat{b}^\epsilon(0,\omega)|^2 = 1 \tag{1.11}$$

which shows that $\hat{a}^\epsilon(L,\omega)$ is uniformly bounded.
This last remark combined with (1.7) and the regularity of $f$ show that the transmitted pulse $((a^\epsilon(L,\sigma))_{-\infty<\sigma<\infty})_{\epsilon>0}$ is a tight family in the space of continuous trajectories equipped with the sup norm. Moreover the finite-dimensional distributions will be characterized by the moments

$$\mathbb{E}[a^\epsilon(L,\sigma_1)^{p_1} \ldots a^\epsilon(L,\sigma_k)^{p_k}]$$

for every real numbers $\sigma_1 < \ldots < \sigma_k$ and every integers $p_1, \ldots, p_k$.
Using the representation (1.7) for each factor $a^\epsilon$, these moments can be written as multiple integrals over $n = \sum_{j=1}^k p_j$ frequencies $\omega_1, \ldots, \omega_n$. The dependency in $\epsilon$ and in the randomness will only appear through the following quantity: $\mathbb{E}[\hat{a}^\epsilon(L,\omega_1) \ldots \hat{a}^\epsilon(L,\omega_n)]$. Our problem is now to find the limit, as $\epsilon$ goes to 0, of these moments for $n$ distinct frequencies. In other words we want to study the limit in distribution of $(\hat{a}^\epsilon(L,\omega_1), \ldots, \hat{a}^\epsilon(L,\omega_n))$ which is an application of classical diffusion-approximation results (see for instance the appendix B of [1] or [11] for the single frequency case).
We define the $n$-dimensional propagator

$$Y_x^\epsilon = Y^\epsilon(x,\omega_1,\omega_2,\ldots,\omega_n) = \begin{bmatrix} Y^\epsilon(x,\omega_1) & & \\ & \ddots & \\ & & Y^\epsilon(x,\omega_n) \end{bmatrix}$$

which satisfies an equation similar to (1.5) with $Y_0^\epsilon = Id_{\mathbb{C}^{2n}}$. Using the complex nature of this propagator we obtain a linear stochastic differential equation satisfied by the limit in distribution denoted by $(Y_x)$ (see [8] for details):

$$\begin{cases} dY_x = PY^\epsilon_x dB_x + \sum_{k=1}^{n}(Q^k Y_x dB^k_x + \tilde{Q}^k Y_x d\tilde{B}^k_x) \\ Y_0 = Id \end{cases} \tag{1.12}$$

where $B, B^1, \ldots, B^n, \tilde{B}^1, \ldots, \tilde{B}^n$ are $2n+1$ independent standard real brownian motions, the stochastic integrals are Ito's type and the matrices $P, Q^1, \ldots, Q^n, \tilde{Q}^1, \ldots, \tilde{Q}^n$ are defined as follows:

$$P = \frac{i\sqrt{\alpha}}{\sqrt{2}} \begin{bmatrix} \omega_1 & & & & \\ & -\omega_1 & & & \\ & & \ddots & & \\ & & & \omega_n & \\ & & & & -\omega_n \end{bmatrix}$$

$Q^k$ and $\tilde{Q}^k$ are respectively equal to:

$$\frac{\sqrt{\alpha}}{2} \begin{bmatrix} 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & \ldots & 0 & \omega_k & 0 & 0 \\ 0 & 0 & \omega_k & 0 & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 \end{bmatrix}, \quad i\frac{\sqrt{\alpha}}{2} \begin{bmatrix} 0 & \ldots & \ldots & & \ldots & \ldots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \ldots & 0 & & \omega_k & 0 & 0 \\ 0 & 0 & -\omega_k & & 0 & \ldots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \ldots & \ldots & & \ldots & \ldots & 0 \end{bmatrix}$$

The positive constant $\alpha$ is the correlation coefficient of the random medium: $\alpha = \int_0^\infty \mathbb{E}(\eta(0)\eta(x))dx$.
The diagonal matrix $P$ is the only one which creates a coupling between distinct frequencies through the brownian motion $B$; each matrix $Q^k$ or $\tilde{Q}^k$ acts only on $Y(x, \omega_k)$.
¿From the stochastic differential equation (1.12) and Ito's formula one can get a stochastic differential equation for $(1/\overline{\alpha}(x, \omega_k), k = 1, \ldots, n)$ denoted by $(\hat{a}(x, \omega_k), k = 1, \ldots, n)$ which is compatible with (1.10) at $x = L$.

$$\begin{aligned} d\hat{a}(x, \omega_k) = & \tfrac{i\sqrt{\alpha}\omega_k}{\sqrt{2}}\hat{a}(x, \omega_k)dB_x - \tfrac{\omega_k^2 \alpha}{2}\hat{a}(x, \omega_k)dx \\ & + F(Y(x, \omega_k))(dB^k_x - id\tilde{B}^k_x) \end{aligned} \tag{1.13}$$

where $F$ is a nonlinear smooth functional and $\hat{a}(0, \omega_k) = 1$.

Another application of Ito's formula gives the following equation for our quantity of interest $\mathbb{E}[\hat{a}^\epsilon(x,\omega_1)\dots\hat{a}^\epsilon(x,\omega_n)]$ denoted by $\phi(x)$:

$$d\phi(x) = -\frac{2\alpha\sum_k \omega_k^2 + \alpha\sum_{k\neq l}\omega_k\omega_l}{4}\phi(x)dx$$

with the initial condition $\phi(0)=1$.
This linear equation has a unique solution but instead of solving it and computing explicitely our moments one can easily see that it is also satisfied by $\tilde{\phi}(x) = \mathbb{E}\left[\prod_k \tilde{a}(x,\omega_k)\right]$ where $\tilde{a}(x,\omega_k)$ is solution of:

$$\text{(1.14)} \qquad \left\{ \begin{array}{ll} d\tilde{a}(x,\omega_k) & = \frac{i\sqrt{\alpha}\omega_k}{\sqrt{2}}\tilde{a}(x,\omega_k)dB_x - \frac{\omega_k^2\alpha}{2}\tilde{a}(x,\omega_k)dx \\ \tilde{a}(0,\omega) & = 1 \end{array} \right.$$

Therefore $\phi(L) = \tilde{\phi}(L)$ and using (1.7) the limit in law $a(L,\sigma)$ is equal to $(2\pi)^{-1}\int e^{-i\omega\sigma}\hat{f}(\omega)\tilde{a}(L,\omega)d\omega$. Solving explicitely the linear stochastic differential equation (1.14), we get the following equality in law:

$$\tilde{a}(x,\omega) = \exp(i\frac{\omega\sqrt{\alpha}}{2}B_x - \frac{\omega^2\alpha}{4}x)$$

Interpreting $\frac{\omega\sqrt{\alpha}}{2}B_L$ as a random phase and $\exp(-\frac{\omega^2\alpha}{4}L)$ as the Fourier transform of the centered gaussian density with variance $\frac{\alpha L}{2}$ denoted by $G_{\frac{\alpha L}{2}}$, one get **the main result of this section**:

$$\text{(1.15)} \qquad a(L,\sigma) = f * G_{\frac{\alpha L}{2}}(\sigma - \sqrt{\frac{\alpha}{2}}B_L)$$

which means that the initial pulse $f$ spreads in a deterministic way through the convolution by a gaussian density and a random gaussian centering appears through the brownian motion $B_L$. We end this section by the following remarks:

- The previous analysis has been done at $L$ fixed. It is not difficult to generalize it to the convergence in distribution of $a^\epsilon(L,\sigma)$ as a process in $\sigma$ and $L$ (see [8] for details). The limit is again given by (1.15) which means that the random centering of the spreaded pulse follows the trajectory of a brownian motion as the pulse travels into the medium.
- In the $\epsilon$-scale, the energy entering the medium at $x=0$ is equal to $\int|f(\sigma)|^2d\sigma$. The energy exiting the medium at $x=L$, in a

coherent way around time $t = L$ in the $\epsilon$-scale, is equal to $\int |f * G_{\frac{\alpha L}{2}}(\sigma)|^2 d\sigma$ which is strictly less than $\int |f(\sigma)|^2 d\sigma$. We may ask the following question: do we have a part of the missing energy exiting the medium in a coherent way somewhere else or at a different time? In other words what is the limit in distribution of $A(L, L + t_0 + \epsilon\sigma)$ for $t_0 \neq 0$ (energy exiting at $x = L$) or $B(0, t_0 + \epsilon\sigma)$ (energy reflected at $x = 0$). A similar analysis shows that these two processes (in $\sigma$) vanish as $\epsilon$ goes to 0 (see [8] for details). This means that there is no other coherent energy in the $\epsilon$-scale exiting the slab $[0, L]$. The next step is to rescale the amplitude of the incoming pulse by a factor $\epsilon^{-\frac{1}{2}}$ so that its energy is of order 1. The analysis requires to study the problem for frequencies depending also on $\epsilon$ (in particular for two frequencies at distance of order $\epsilon$). This is done in [5] and [1].

**1.2. The general one-dimensional case.** Going back to the conservation laws (1.1), we now suppose that the coefficients $\rho(x)$ and $K(x)$ vary on the macroscopic scale and that the fluctuations are present in both coefficients:

$$\left\{ \begin{array}{ll} \rho(x) & = \rho_0(x)(1 + \eta(\frac{x}{\epsilon^2})) \\ \frac{1}{K(x)} & = \frac{1}{K_0(x)}(1 + \nu(\frac{x}{\epsilon^2})) \end{array} \right.$$

We introduce the acoustic impedance $\zeta$ and the acoustic speed $c$ defined by $\zeta(x) = \sqrt{\rho_0(x)K_0(x)}$ and $c(x) = \sqrt{K_0(x)\rho_0(x)^{-1}}$. The right and left going waves are given by $A = \zeta^{-1/2}p + \zeta^{1/2}u$ and $B = -\zeta^{-1/2}p + \zeta^{1/2}u$ which satisfy:

$$\begin{array}{rcl} \frac{\partial}{\partial x} \left[ \begin{array}{c} A \\ B \end{array} \right] & = & \frac{1}{c(x)} \left( \left[ \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right] + \left[ \begin{array}{cc} -m(\frac{x}{\epsilon^2}) & -n(\frac{x}{\epsilon^2}) \\ n(\frac{x}{\epsilon^2}) & m(\frac{x}{\epsilon^2}) \end{array} \right] \right) \frac{\partial}{\partial t} \left[ \begin{array}{c} A \\ B \end{array} \right] \\ & & + \Lambda(x) \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \left[ \begin{array}{c} A \\ B \end{array} \right] \end{array}$$

where $m = \frac{1}{2}(\eta + \nu)$, $n = \frac{1}{2}(\eta - \nu)$ and $\Lambda = \frac{d}{dx}(\ln \zeta^{\frac{1}{2}})$ and with boundary conditions (1.3).
Introducing the travel time $\tau$ in the macroscopic medium defined by $\tau(x) = \int_0^x \frac{dy}{c(y)}$, we can perform the centering and rescaling:

$$\left\{ \begin{array}{ll} a^\epsilon(x, \sigma) & = A(x, \tau(x) + \epsilon\sigma) \\ b^\epsilon(x, \sigma) & = B(x, -\tau(x) + \epsilon\sigma) \end{array} \right.$$

so that the transmitted pulse under study is $A(L, \tau(L) + \epsilon\sigma)$. The representation (1.7) holds for $(\hat{a}^\epsilon, \hat{b}^\epsilon)$ solution of:

$$\frac{d}{dx}\begin{bmatrix}\hat{a}^\epsilon\\ \hat{b}^\epsilon\end{bmatrix} = \frac{i\omega}{\epsilon c(x)}\begin{bmatrix} m(\frac{x}{\epsilon^2}) & n(\frac{x}{\epsilon^2})e^{-2i\omega\frac{\tau(x)}{\epsilon}} \\ -n(\frac{x}{\epsilon^2})e^{2i\omega\frac{\tau(x)}{\epsilon}} & -m(\frac{x}{\epsilon^2})\end{bmatrix}\begin{bmatrix}\hat{a}^\epsilon\\ \hat{b}^\epsilon\end{bmatrix}$$
$$+\omega\Lambda(x)\begin{bmatrix} 0 & e^{-2i\omega\frac{\tau(x)}{\epsilon}} \\ e^{2i\omega\frac{\tau(x)}{\epsilon}} & 0\end{bmatrix}\begin{bmatrix}\hat{a}^\epsilon\\ \hat{b}^\epsilon\end{bmatrix}$$

with the boundary conditions (1.6). The method presented in the previous section can be generalized to our situation (we refer to [8] for details). The matrices $P$, $Q^k$, $\tilde{Q}^k$ are now $x$ dependent and there is a drift term added to (1.12). Introducing the correlation coefficients:

$$\begin{cases} \alpha_m = \int_0^\infty \mathbb{E}(m(0)m(x))dx \\ \alpha_n = \int_0^\infty \mathbb{E}(n(0)n(x))dx \end{cases}$$

**the main result of this section** is as follows: $A(L, \tau(L)+\epsilon\sigma) = a^\epsilon(L,\sigma)$ converges in distribution, $\epsilon$ going to 0, as a continuous process in $-\infty < \sigma < +\infty$ and $L \leq 0$, to $a(L,\sigma)$ given by:

$$a(L,\sigma) = f * G_L(\sigma - \sqrt{2\alpha_m}Z_L) \tag{1.16}$$

where $G_L$ is the centered gaussian density with variance $2\alpha_n\int_0^L \frac{dy}{c^2(y)}$ and $Z_L$ is a random gaussian centering given by $\sqrt{2\alpha_m}\int_0^L \frac{dB_y}{c(y)}$ where $B$ is a standard brownian motion. The interpretation of this result and the remarks following (1.15) are the same as in the previous section.

**1.3. The three-dimensional layered case.** In order to stay in a reasonable length we shall not give any details in this section (we refer for that to a work in preparation [6] ). The situation is the same as the one studied in [1] : acoustic waves propagating in a three-dimensional randomly layered medium; the randomness occurs between depth $z = 0$ and $z = -L$, depends only upon the depth $z$ and varies on a small scale $\epsilon$. For simplicity we suppose that there is no macroscopic variations in the coefficients and only fluctuations in $K(x)^{-1} = 1 + \nu(\frac{z}{\epsilon^2})$. A point source is placed above the medium at $x = y = 0, z = z_s \geq 0$. We assume that $-L$ is above all the turning points. Because of the transverse propagation the downgoing wave is not the quantity to work with; instead, we study the pressure at depth $-L$, at a lateral distance $r = \sqrt{x^2+y^2}$ from the point source and around the time of direct arrival $t = \sqrt{r^2+L^2}$:

$$p^\epsilon(L,r,\sigma) = p(x,y,z=-L,t=\sqrt{r^2+L^2}+\epsilon\sigma)$$

We perform a Fourier transform in time and in the tranverse variables $(x, y)$ and we denote by $\omega$ and $(h, k)$ the corresponding variables in the frequency domains: with the notation $\tau = \tau(h, k) = \sqrt{1 - h^2 - k^2}$, the pressure $p^\epsilon(L, r, \sigma)$ admits the following integral representation:

$$\frac{-1}{16\pi^3\epsilon}\int\int\int e^{-\frac{i\omega}{\epsilon}(\epsilon\sigma+\sqrt{r^2+L^2}-hx-ky-\tau L)}C(\omega, h, k, 0)\hat{f}(\omega)\omega^2 dhdkd\omega$$

where the integral over $(h, k)$ is taken on a disk $h^2 + k^2 \leq R^2$ of radius $R$ strictly less than 1 since we stay above the turning points and $C(\omega, h, k, z)$ is obtained as the solution of the following system of equations:

$$\begin{cases} \dfrac{dC^\epsilon}{dz} & = \frac{i\omega}{2\epsilon\tau}\nu(\frac{z}{\epsilon^2})(1 - \Gamma^\epsilon e^{-\frac{2i\omega\tau z}{\epsilon}})C^\epsilon \\ \dfrac{d\Gamma^\epsilon}{dz} & = \frac{i\omega}{2\epsilon\tau}\nu(\frac{z}{\epsilon^2})(e^{\frac{2i\omega\tau z}{\epsilon}} - 2\Gamma^\epsilon + (\Gamma^\epsilon)^2 e^{-\frac{2i\omega\tau z}{\epsilon}}) \end{cases}$$

with the conditions $\Gamma^\epsilon(\omega, h, k, z = -L) = 0$ and $C^\epsilon(\omega, h, k, z = -L) = 1$. The second equation is the Ricatti equation for the reflexion coefficient $\Gamma^\epsilon$ and the first equation is linear in $C^\epsilon$.
Combining the method of moments presented in the first section, an appropriate stationary phase result and an approximation-diffusion result we are able to prove that $p^\epsilon(L, r, \sigma)$ converges in law as a continuous process in $\sigma, L$ and $r$. The limit is given by:

$$p(L, r, \sigma) = \frac{-1}{4\pi\sqrt{L^2 + r^2}} f * \tilde{G}_{L,r}(\sigma - \sqrt{\frac{\alpha}{2}(1 + \frac{r^2}{L^2})}B_L)$$

where $\tilde{G}_{L,r}$ is the derivative of the centered gaussian density with variance $\frac{\alpha}{2}(1 + \frac{r^2}{L^2})L$, $B$ is a standard brownian motion and the positive coefficient $\alpha$ is equal to $\int_0^\infty \mathbb{E}(\nu(0)\nu(x))dx$. This gives a complete description of the law, as $\epsilon$ goes to 0, of the pressure field at the bottom of the random slab and in particular, it shows that the randomness depends only on the depth $L$ through the brownian motion $B$.

**2. Parabolic and white noise approximation.** The parabolic or forward scattering approximation has been used extensively in the study of wave propagation. In this section we present the results of [3] where it is shown that this approximation can be combined with a gaussian white noise approximation for waves propagating in a random medium. This will be the case for weakly fluctuating media and in the high frequencies regimes. The limiting distribution of the wave field is characterized as the

unique solution of a random Schrödinger equation studied by Dawson and Papanicolaou [9].
We consider the reduced wave equation for the scalar field $\Psi(r), r \in \mathbb{R}^3$:

$$\nabla^2\Psi + k^2 n^2(r)\Psi = 0 \tag{2.1}$$

where $k$ denotes the free space wave number and $n(r)$ a random index of refraction given by:

$$\begin{cases} n^2(r) = 1 + \beta\eta(\alpha r_1, \alpha r_2, r_3) & \text{for } 0 \le r_3 \le L \\ n^2(r) = 1 & \text{elsewhere} \end{cases} \tag{2.2}$$

The random field $\eta$ is assumed to be real bounded and centered with mixing properties in the $r_3$-direction; $\beta$ is a positive parameter measuring the size of the fluctuations such that $\beta|\eta|$ is bounded by a constant less than 1; $\alpha$ is a positive scale ratio between the $r_3$-direction and the $(r_1, r_2)$ othogonal directions and $L$ is a length scale corresponding to a distance of propagation in the $r_3$-direction.
Equation (2.1) is to be solved on the half-space $\{r_3 \ge 0\}$ with the following boundary conditions:

$$\begin{cases} \Psi(r_1, r_2, 0) = \Psi_0(\alpha r_1, \alpha r_2) \\ \text{radiation condition on } r_3 > L \end{cases} \tag{2.3}$$

where $\Psi_0$ is a smooth function which has a bounded Fourier transform $\hat{\Psi}_0$ with compact support; our radiation condition corresponds to the absence of left going wave on $r_3 > L$.
We shall exhibit various regimes for which the elliptic problem (2.1) with boundary conditions (2.3) is expected to converge to a stochastic parabolic problem with an initial condition at $r_3 = 0$. We are not concerned by the wellposedness of problem (2.1),(2.3): this will be garantied in the models studied in [3] and [2].
We are interested in the solution of (2.1) which propagates mainly in the $r_3$-direction. In order to center our problem, we define the field $u(r)$ by:

$$\Psi(r) = e^{ikr_3}u(r) \tag{2.4}$$

which satisfies:

$$\left(\frac{\partial^2}{\partial r_1^2} + \frac{\partial^2}{\partial r_2^2}\right)u + \frac{\partial^2 u}{\partial r_3^2} + 2ik\frac{\partial u}{\partial r_3} + k^2(n^2(r) - 1) = 0 \tag{2.5}$$

with the boundary conditions corresponding to (2.3).
Introducing a new positive parameter $\epsilon$, we perform the rescaling $(r_1, r_2) = (x_1\alpha^{-1}, x_2\alpha^{-1})$ and $r_3 = t\epsilon^{-2}$, so that, using (2.2) and denoting $(x_1, x_2)$ by $x$ and $\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ by $\Delta$, (2.5) becomes:

$$i\frac{\partial u}{\partial t} + \frac{\epsilon^2}{2k}\frac{\partial^2 u}{\partial t^2} + \frac{\alpha^2}{2k\epsilon^2}\Delta u + \frac{k\beta}{2\epsilon^2}\eta(\frac{t}{\epsilon^2}, x)u = 0 \tag{2.6}$$

with the boundary conditions: $u(0, x) = \Psi_0(x_1, x_2)$ also denoted $u_0(x)$ and a radiation condition on $t > \epsilon^2 L$.
Let us assume that $L$ is of order $\epsilon^{-2}$ such that $L = T/\epsilon^2$. Equation (2.6) is to be solved on $[0, T]$.
We are interested in the parabolic and white noise approximation of this equation: the parabolic or forward scattering approximation consists in neglecting the backscattered waves, namely the second derivative with respect to $t$, keeping of order 1 the second derivative in the orthogonal direction $x$. This approximation is commonly made in the litterature (Ishimaru [13], Dawson-Papanicolaou [9], Nair-White [16]). The second approximation is the white noise approximation: it consists in approximating $\frac{k\beta}{2\epsilon^2}\eta(\frac{t}{\epsilon^2}, x)$ by a white noise in the $t$-variable.
The parabolic approximation is of great interest because solving directly (2.5) requires a large amount of computations. It has been justified when the angle of propagation is small, far from the source and for small variations in the range of the sound speed (see for example Bamberger and al [4] ). On the other hand, the white noise approximation enables us to use stochastic calculus on the limiting field in order, for instance in our linear case, to close the equations for the moments of the solution.
We now reformulate our problem (2.6) in the regimes described above. For the parabolic approximation we impose $\frac{\alpha^2}{2k\epsilon^2}$ to be of order 1 and $\frac{\epsilon^2}{2k}$ to be small. For the white noise approximation we impose $\frac{k\beta}{2\epsilon^2}$ to be of order $\epsilon^{-1}$.
Setting $\frac{\epsilon^2}{2k} = \epsilon^p$ for some parameter $p$, one gets: $k = \frac{1}{2}\epsilon^{2-p}$, $\beta = 4\epsilon^{p-1}$ and $\alpha = \epsilon^{\frac{4-p}{2}}$.
The condition $1 \leq p \leq 4$ will ensure that $\beta$ and $\alpha$ are small or at most of order 1. It will turn out that these approximations will be valid only in the high frequencies regimes which correspond to $p > 2$. So we assume:

$$2 < p \leq 4 \tag{2.7}$$

With $p$ satisfying (2.7), our problem is now only $\epsilon$ dependent and it can be rewritten as:

$$i\frac{\partial u^\epsilon}{\partial t}+\epsilon^p\frac{\partial^2 u^\epsilon}{\partial t^2}+\Delta u^\epsilon+\frac{1}{\epsilon}\eta(\frac{t}{\epsilon^2},x)u^\epsilon=0 \tag{2.8}$$

with the boundary conditions:

$$\begin{cases} u^\epsilon(0,x)=u_0(x) \\ \text{radiation condition on } t>T \end{cases} \tag{2.9}$$

In the applications, $\beta$ and $\alpha$ will be given; we are interested in the cases where $\beta$ is small and $\alpha$ at most of order 1 so that our parameter $\epsilon$ is of order $(\alpha^2\beta)^{1/3}$, $k$ is of order $(\frac{\alpha}{\beta})^{2/3}$. For instance, the case $\alpha=1$, $\beta$ small ($p=4$) corresponds to the model used in [16].
Denoting by $W(t,x)$ the infinite dimensional brownian motion obtained as the limit in law of the process $W^\epsilon(t,x)=\frac{1}{\epsilon}\int_0^t\eta(\frac{s}{\epsilon^2},x)ds$ and assuming $\eta$ to be stationary in the $x$-direction, $u^\epsilon(x,t)$ should converge to a solution of the following stochastic partial differential equation:

$$\begin{aligned} u(t,x)=u_0(x) \quad & +i\int_0^t\Delta u(s,x)ds+i\int_0^t u(s,x)dW(s,x) \\ & -\tfrac{1}{2}\sigma^2\int_0^t u(s,x)ds \end{aligned} \tag{2.10}$$

where the stochastic integral is an Itô integral and $-\frac{1}{2}\sigma^2 u$ is a Stratonovich correction with $\sigma^2=\int_0^\infty \mathbb{E}(\eta(s,0)\eta(0,0))ds$.
This random Schrödinger equation has been studied by Dawson-Papanicolaou [9]: they proved the uniqueness of the solution. One can observe that such a limiting result requires to take a limit in an elliptic problem (2.8) with boundary conditions (2.9) and to obtain a parabolic problem with an initial condition; in particular the radiation condition in (2.9) disappears in the limit.
We consider $u^\epsilon(t,x)$ and $u(t,x)$ as continuous stochastic processes in $t\geq 0$ taking their values in the space $\mathbb{L}^2_{\mathbb{C}}(\mathbb{R}^2)$ equipped with its weak topology. **The main result of this section** is that $u^\epsilon(t,x)$, solution of (2.8) with boundary conditions (2.9), converges in law, as $\epsilon$ goes to 0, to $u(t,x)$, solution of (2.10).
Such a result is derived by introducing the following parabolic equation:

$$i\frac{\partial v^\epsilon}{\partial t}+\Delta v^\epsilon+\frac{1}{\epsilon}\eta(\frac{t}{\epsilon^2},x)v^\epsilon=0 \tag{2.11}$$

with the initial condition $v^\epsilon(0,x) = u_0(x)$ and proving first that $v^\epsilon(t,x)$ converges in law to $u(t,x)$. We then expand $u^\epsilon(t,x)$ as an infinite series involving $u_0(x)$, $\frac{\partial u^\epsilon}{\partial t}(0,x)$ and their derivatives. This gives a strong solution of equation (2.8) and requires strong uniform estimates on $\frac{\partial u^\epsilon}{\partial t}(0,x)$ and its derivatives. We then show that $u^\epsilon(t,x)$ has the same limit as $v^\epsilon(t,x)$. The particular case of a stratified model is treated in [3] and the general case in the work in progress [2].

**3. Transmission through a one-dimensional weakly nonlinear random medium.** We consider a one-dimensional Helmholtz equation with a small random perturbation and study the problem of the decay of the transmission coefficient for large lengths with fixed output. It is well known that in the linear case we have exponential localization, while with a strong nonlinear term we have a polynomial behaviour of the transmittivity (see [14]). Our aim is to show that we still have exponential localization with a weak nonlinear term.

We study the propagation of monochromatic waves through a finite, nonlinear, disordered slab $[0, L]$. The time harmonic scalar field $U$ satisfies the equation :

$$U_{xx} + k^2 n^2(x, |U|^2)U = 0, \tag{3.1}$$

where $k$ is the free wave number, and $n(x, |U|^2)$ is the index of refraction of the medium. Outside the slab $[0, L]$ space is free and the index of refraction is equal to one. We assume that a plane wave of amplitude $A$ is coming from the right, so that there are an incident wave from the right and a reflected wave for $x \geq L$, and a transmitted wave for $x \leq 0$. Therefore the wave $U$ may be presented in the following form, outside the slab $[0, L]$ :

$$\begin{cases} U(x) = A(e^{-ik(x-L)} + Re^{ik(x-L)}) & x \geq L, \\ U(x) = ATe^{-ikx} & x \leq 0, \end{cases} \tag{3.2}$$

where $R$ et $T$ are respectively the reflection and transmission coefficients. They depend on the length of the slab $L$ and on the amplitude of the incoming wave $A$ because of nonlinearity. The model for the medium inside the slab is deduced from some optical media, whose indices of refraction are affected by the intensity of light and by random inhomogeneities. Therefore we shall take the index of refraction to have the form :

$$n(x, |U|^2) = 1 + \varepsilon m(x) + \varepsilon^a \tilde{\gamma} |U|^2,$$

where $\varepsilon$ is a small parameter, $\tilde{\gamma}$ and $a$ are positive constants and $m$ is a centered Markov process which satisfies mixing conditions.

**Remark.** We could have considered the nonlinear stationary Schrödinger equation with a random potential :

$$-\psi_{xx} + V(x)\psi - \alpha|\psi|^2\psi = k^2\psi.$$

However the scalings we are about to describe have to be adapted (see [12] ).

There are three length scales in our problem corresponding to the correlation radius of the perturbation, the length of the slab and the wavelength. Moreover there are two intensity scales corresponding to the intensity of the incoming wave and the amplitude of the perturbation. We use a small parameter $\varepsilon$ in order to establish ratio between these scales, and then we shall study the limit as $\varepsilon$ tends to 0.

The limit we are going to deal with corresponds to classical situations in optics (cf [14] ). We assume that the correlation radius of the perturbation $m$ and the wavelength are of the same order and are small compared to the length of the slab $L$. Therefore we consider that the correlation length and the wavelength are of order $\varepsilon^2$ while $L$ is of order one. We assume also that the random perturbation is weak, of order $\varepsilon$, otherwise the problem has no limit as $\varepsilon$ tends to 0. With such a scaling $\varepsilon m(x)$ has a behaviour similar to that of a white noise when $x \sim \varepsilon^2$. Knapp, Papanicolaou and White in [14] studied the limit corresponding to this scaling when the size of the nonlinearity is of order one, i.e. the case when $a = 0$. We shall consider a nonlinearity of order $\varepsilon^a$ for some real $a > 0$ and show that the two results are radically different.

We aim at analysing the scattering problem which consists of the equation (3.1) with the boundary conditions (3.2). We introduce the normalized field $u$ such that $U(x) = Au(x)$. If we introduce the intensity of the incoming wave $w = |A|^2$, then the equations (3.1,3.2) can be rewritten as :

$$\begin{cases} u_{xx} + k^2(1 + \varepsilon m(x) + \varepsilon^a \tilde{\gamma} w |u|^2)u = 0, \\ -\dfrac{i}{k} u_x(0) + u(0) = 0, \\ \dfrac{i}{k} u_x(L) + u(L) = 2. \end{cases} \tag{3.3}$$

The term $\varepsilon^a \tilde{\gamma} w$ governs the strength of the nonlinearity of the system.

The boundary value problem (3.3) can be replaced by an initial value problem parametrized by the output intensity $w_0 = |T|^2 w$. Let us define $q$, $\theta$ by $u(x) = \frac{q(x)}{|T|} e^{-i\theta(x)}$. They satisfy

$$\begin{cases} q_x = p, & q(0) = 1, \\ p_x = k^2 \left( \frac{1}{q^3} - (1 + \varepsilon m(x))q - \varepsilon^a \gamma q^3 \right), & p(0) = 0, \end{cases} \tag{3.4}$$

where $\gamma = \tilde{\gamma} w_0$. The square modulus of the transmission coefficient $|T|^2$ can be expressed as :

$$|T|^2(L) = \frac{2k^2}{E(L) + k^2 - \frac{1}{4} k^2 \gamma \varepsilon^a q^4(L)},$$

where $E$ is the energy given by :

$$E(L) = \frac{1}{2} p^2(L) + V^\varepsilon(q(L)), \quad V^\varepsilon(q) = \frac{k^2}{2}(q^2 + \frac{1}{q^2} + \frac{1}{2} \varepsilon^a \gamma q^4). \tag{3.5}$$

In order to explicit the periodic structure of the fast varying components of the variable $q$, $p$ $E$, we introduce the action angle variables. The action $I$ is defined as a function of the energy $E$ by

$$I^\varepsilon(E) = \frac{1}{2\pi}\oint p dq = \frac{1}{2\pi}\oint \sqrt{2E - 2V^\varepsilon(q')}dq'.$$

Indeed, if you fix $E$, then the motion described by (3.5) is periodic, with period $\pi^\varepsilon(E) = \left(\frac{1}{2\pi}\oint \frac{dq'}{\sqrt{2E-2V^\varepsilon(q')}}\right)^{-1}$.
The angle $\phi$ is defined as a function of $I$ and $q$ by

$$\phi^\varepsilon(E,q) = -\int^q \frac{\partial p}{\partial I}dq = -\pi^\varepsilon(E)\int^q \frac{dq'}{\sqrt{2E - 2V^\varepsilon(q')}}.$$

The transformation $(E,q) \to (I,\phi)$ can be inverted to give the functions $E^\varepsilon(I)$ and $Q^\varepsilon(I,\phi)$. Thus, after rescaling $x = \frac{t}{\varepsilon^2}$ we will study the random processes $I^\varepsilon(\frac{t}{\varepsilon^2})$ and $\phi^\varepsilon(\frac{t}{\varepsilon^2})$ which are solution of the differential equations :

$$\text{(3.6)} \qquad \left\{ \begin{array}{l} \dfrac{dI^\varepsilon}{dt}(\dfrac{t}{\varepsilon^2}) = \dfrac{1}{\varepsilon}m(\dfrac{t}{\varepsilon^2})k^2 h^\varepsilon_\phi(I^\varepsilon,\phi^\varepsilon), \\ \dfrac{d\phi^\varepsilon}{dt}(\dfrac{t}{\varepsilon^2}) = -\dfrac{1}{\varepsilon^2}\omega^\varepsilon(I^\varepsilon) - \dfrac{1}{\varepsilon}m(\dfrac{t}{\varepsilon^2})k^2 h^\varepsilon_I(I^\varepsilon,\phi^\varepsilon), \end{array} \right.$$

where $h^\varepsilon(I,\phi) = \frac{1}{2}Q^{\varepsilon 2}(I,\phi)$ and $\omega^\varepsilon(I) = \pi^\varepsilon(E^\varepsilon(I))$ are smooth functions and $h^\varepsilon$ is periodic with respect to $\phi$. We aim at finding a limit in law for the process $(I^\varepsilon(\frac{t}{\varepsilon^2}),\phi^\varepsilon(\frac{t}{\varepsilon^2}))_{t\geq 0}$. The equations (3.6) take place in the scales of the approximation-diffusion. However we cannot use the classical theorem because $\omega^\varepsilon$ and $h^\varepsilon$ explicitly depend on $\varepsilon$. Then the work can be divided in three steps. First we prove the existence of the expansions of $\omega^\varepsilon$ and $h^\varepsilon$ in powers of $\varepsilon^a$. Secondly we apply a multi-scaled diffusion-approximation theorem (see [12] ) to prove the existence of a limit Markov process $(I(t),\psi(t))_{t\geq 0}$ characterized by its generator. Finally we get back to the variables $(E,q)$, and we identify the limit in law for $|T^\varepsilon|^2$. Then we obtain **the main result of this section** : The weak limit of $|T^\varepsilon|^2$ is the same as the one of the square modulus of the transmission coefficient in the linear case. In particular,

$$\lim_{L\to\infty}\frac{1}{L}\ln \mathbb{E}[|T|^2(L)] = -\alpha(k)k^2, \quad \alpha(k) = \int_0^\infty \cos(2ks)\mathbb{E}[m(0)m(s)]ds.$$

**4. Amplification of incoherent light.** We aim at studying the interaction between nonlinear and random terms in the propagation of a Gaussian incoherent field through a nonlinear amplifier medium. The medium, which draws power from a source other than the input signal, provides the pulse with energy. Our first task is to study this phenomenon called amplification. However, the medium is not perfectly linear, and we have to consider that the index of refraction is slightly nonlinear. Then, because of self phase modulation, we shall see that it induces spectral broadening on the one hand, and that it affects the amplification on the other one.

The results of this paper seem to agree with experimental observations concerning the amplification and the spectral broadening of an incoherent laser beam. In particular a saturation of the output intensity is observed while the transverse effects and the variation in the population inversion are negligeable.

We consider a plane wave incoming from the left $z < 0$, with the form $E_0(t)$ at $z = 0$. We assume that $E_0$ is a complex Gaussian field with correlation function $K_0(t) =< E_0(0)E_0^*(t) >$ given by $K_0(t) = A_0^2 f(t)$. The angle brackets refer to the expectation with respect to the statistics of $E_0$. $A_0$ is the initial amplitude of the electric field ($A_0^2$ is the initial intensity). The width of the correlation function $f$ will be denoted by $T_c$. We shall call it coherence time and define it by $\int_{-T_c}^{T_c} f(t)dt = \frac{1}{2}\int_{-\infty}^{\infty} f(t)dt$.

We shall neglect the transverse aspect and consider that the medium is not gradually failing during the amplification, i.e. that the population inversion $n$ is constant.

During its propagation, the electric field is subject to a self phase modulation, induced by a nonlinear index of refraction :

$$\left\{ \begin{array}{l} E = E(z,T), \quad T = t - \dfrac{z}{v_g}, \\ -2i\dfrac{\partial E}{\partial z} + \gamma |E|^2 E = P, \end{array} \right. \tag{4.1}$$

$T$ is the reduced time at position $z$, $v_g$ is the group velocity and $P$ is the polarization of the amplifier medium, whose evolution is governed by :

$$\varepsilon \frac{\partial P}{\partial t} + P = -inE, \tag{4.2}$$

We shall assume that the characteristic time of the evolution of the polarization is much shorter than the coherence time of the initial field, i.e. that $\varepsilon \ll T_c$.

**4.1. Saturation of the amplification.** On the one hand the self phase modulation does not affect the growth of the intensity if we consider the limit case $\varepsilon = 0$ (i.e. $P = -inE$). On the other hand the evolution of the polarization (4.2) has only a negligible influence on the intensity if we set aside the self phase modulation (typically $nz \simeq 10$). Indeed, in such approximations, the average output intensity is close to the "expected" output intensity $\bar{I}_e = A_0^2 e^{nz}$.

$$I_{\varepsilon,\gamma=0} = \bar{I}_e, \quad I_{\varepsilon=0,\gamma} \simeq \bar{I}_e \left( 1 - \left(\frac{\varepsilon}{T_c}\right)^2 (nz) + O\left(\frac{\varepsilon}{T_c}\right)^4 \right).$$

If we take into account both phenomena, then the average intensity is

$$(4.3)\quad I_{\varepsilon,\gamma} \simeq \bar{I}_e \left(1 - \left(\frac{\varepsilon}{T_c}\right)^2 nz - \left(\frac{\varepsilon}{T_c}\frac{\gamma}{2n}\bar{I}_e\right)^2 \beta(nz) + O\left(\frac{\varepsilon}{T_c}\right)^3\right),$$

where $\beta(x) = 2 - 8e^{-x} + 4xe^{-2x} + 6e^{-2x}$, and the term $O\left(\frac{\varepsilon}{T_c}\right)^3$ is a sum of terms of the type $\left(\frac{\varepsilon}{T_c}\frac{\gamma}{2n}\bar{I}_e\right)^j$, $j \geq 3$, and some others which are negligible in front of these, like $\left(\frac{\varepsilon}{T_c}nz\right)^j$, $j \geq 3$. The new corrective term depends on the expected output intensity $\bar{I}_e$, which we want to be very large. As a consequence, we get **the main result of this section** : When the expected output intensity $\bar{I}_e = A_0^2 e^{nz}$ reaches values of order $\left(\frac{\varepsilon}{T_c}\frac{\gamma}{2n}\right)^{-1}$, then the amplification is stopped.

**4.2. Spectral broadening.** The correlation function at position $z$ is $K(z,T) =< E(z,u)E^*(z,u+T) >$. It is greatly affected by the self phase modulation

$$(4.4)\qquad K_{\varepsilon=0,\gamma}(T) = \frac{\bar{I}_e f(T)}{\left(1 + \left(\frac{\gamma}{2n}\bar{I}_e\right)^2 \alpha(nz)^2(1 - f(T)^2)\right)^2},$$

where $\alpha(x) = 1 - e^{-x}$. If we take into account the evolution of the polarization (4.2), then we get the same type of expansion than the one of the average intensity. We find that $K_{\varepsilon,\gamma}(T)$ is equal to

$$K_{\varepsilon=0,\gamma}(T) \times \left(1 + \left(\frac{\varepsilon}{T_c}\right)^2 nzg(T) + \left(\frac{\varepsilon}{T_c}\frac{\gamma}{2n}\bar{I}_e\right)^2 \lambda(nz)h(T) + O\left(\frac{\varepsilon}{T_c}\right)^3\right),$$

where $\lambda(x) \simeq 2x^2$. The correlation function $K(T)$ becomes more and more concentrated near $T = 0$ when $\bar{I}_e$ increases. It means that the coherence time decreases. In fact, we can exhibit (see [10]) a relation between $\bar{I}_e$ and the spectral width (which is proportional to the inverse of the coherence time). In particular, we can notice that the spectral width goes on increasing when $\bar{I}_e$ reaches values of order $\left(\frac{\varepsilon}{T_c}\frac{\gamma}{2n}\right)^{-1}$, while the output intensity is saturated.

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