SIAM J. APPL. MATH.
Vol. 68, No. 6, pp. 1574–1599

EFFECTIVE TRANSPORT EQUATIONS AND ENHANCED BACKSCATTERING IN RANDOM WAVEGUIDES*

JOSSELIN GARNIER† AND KNUT SØLNA‡

Abstract. In this paper we derive a general system of transport equations for the moments of reflected and transmitted mode amplitudes in a randomly perturbed waveguide, in a regime where backscattering is significant. The derivation is based on a limit theorem for the system of coupled differential equations for the mode amplitudes, in the limit where the amplitude of the random fluctuations of the medium is small, the correlation lengths in the transverse and longitudinal directions are of the same order of the wavelength, and the waveguide is long. Using this system we derive several results in specific regimes, including the enhanced backscattering phenomenon for the reflected wave: when an incoming monochromatic wave with a specific incidence angle is present, the mean reflected power has a local maximum in the backward direction twice as large as the mean reflected power in the other directions.

Key words. acoustic waveguides, random media, asymptotic analysis

AMS subject classifications. 76B15, 35Q99, 60F05

DOI. 10.1137/070694909

1. Introduction.

This paper is devoted to the analysis of wave propagation in a random waveguide. We use a separation of scales technique introduced by Papanicolaou and his co-authors. Although this technique was originally introduced for wave propagation in one-dimensional random media [1], it has been shown recently that it is possible to extend the technique to three-dimensional random media in the context of waveguides [5, 7, 6]. By writing the coupled mode equations for the complex mode amplitudes, diffusion approximation theorems can be applied, leading to differential equations driven by Brownian motions whose solutions are Itô diffusion processes. In [5, 6, 7] the analysis was restricted to the forward scattering approximation, where the conversion from forward-going to backward-going modes is neglected. This regime is characterized by the equipartition of energy: the mean transmitted mode powers become uniformly distributed when the waveguide is long enough. In this paper, we revisit this analysis in the general case and take into account backscattering. We derive a system of transport equations for the moments of the reflected and transmitted mode amplitudes in the regime where the fluctuations of the random medium have a small amplitude and a correlation length of the same order as the typical wavelength. This allows us to exhibit the enhanced backscattering phenomenon: when a monochromatic input mode is applied, the mean reflected mode powers become uniformly distributed, except for the mode corresponding to the backward direction, where the mean reflected power is twice the mean power of the other modes. This phenomenon, also known as weak localization, is well referenced in the physical literature and it has been observed in several experimental contexts, such as in optics with

*Received by the editors June 20, 2007; accepted for publication (in revised form) February 29, 2008; published electronically June 6, 2008. This work was supported by ONR grants N00014-02-1-0089 and N00014-05-1-0442.
http://www.siam.org/journals/siap/68-6/69490.html
†Laboratoire de Probabilités et Modèles Aléatoires & Laboratoire Jacques-Louis Lions, Université Paris VII, 2 Place Jussieu, 75251 Paris Cedex 5, France (garnier@math.jussieu.fr). This author was supported by ANR.
‡Department of Mathematics, University of California, Irvine, CA 92697 (ksolna@math.uci.edu). This author was supported by NSF grant DMS0307011 and the Sloan Foundation.

powder suspensions [17, 15], with biological tissues [18], and with ultracold atoms [11] as well as in acoustics [14]. The physical analysis of the weak localization is based on diagrammatic expansions [16], where interference effects between direct and reverse paths play a crucial role. Here we give a mathematical derivation of this phenomenon by an asymptotic analysis in the context of random waveguides. We also compute the second moments of the reflected mode powers and show that these quantities are not statistically stable in the sense that their fluctuations are of the same order as their mean values. This means that it is necessary to average the reflected power to detect the enhanced backscattering. This point was already mentioned in the physical literature, and we give here a quantitative analysis of this phenomenon.

2. Propagation in a random waveguide. We consider wave propagation in a waveguide where the medium parameters have small random perturbations. Many modern applications involve propagation in waveguides [3, 4, 13]. We will here describe the problem in a scaling regime where the radius of the waveguide is of the order of a few wavelengths and with the medium parameters varying randomly in the longitudinal and transversal directions with a correlation length on the order of the wavelength. This scaling regime was also considered in [5, 7, 6]. The analysis could be generalized to other scaling limits whenever diffusion approximation theorems can be applied.

We consider linear acoustic waves propagating in three spatial dimensions:

$$\rho(\mathbf{x},z)\frac{\partial \mathbf{u}}{\partial t} + \nabla p = \mathbf{0}, \quad \frac{1}{K(\mathbf{x},z)}\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{u} = 0 \quad \text{for } \mathbf{x}\in\mathcal{D} \text{ and } t,z\in\mathbb{R}, \tag{2.1}$$

where p is the pressure field, $\mathbf{u}$ is the velocity field, ρ is the density of the medium, K is the bulk modulus, and $(\mathbf{x},z) = (x,y,z)$ stands for the space coordinates. The cross section of the waveguide is denoted by $\mathcal{D}$, and we shall use Dirichlet boundary conditions

$$p(t,\mathbf{x},z) = 0 \qquad \text{for } \mathbf{x}\in\partial\mathcal{D} \text{ and } z\in\mathbb{R}. \tag{2.2}$$

The direction of propagation along the waveguide axis is z and the transverse coordinates are denoted by $\mathbf{x}\in\mathcal{D}$. The random part of the waveguide occupies the region $z\in[0,L/\varepsilon^2]$ and is embedded in between two homogeneous waveguide sections. Inside the perturbed waveguide the bulk modulus is randomly varying, and we assume for simplicity that the density is homogeneous:

$$\frac{1}{K(\mathbf{x},z)} = \begin{cases} \frac{1}{\bar K}\left(1+\varepsilon\nu(\mathbf{x},z)\right) & \text{for} \quad \mathbf{x}\in\mathcal{D}, \quad z\in[0,L/\varepsilon^2], \\ \frac{1}{\bar K} & \text{for} \quad \mathbf{x}\in\mathcal{D}, \quad z\in(-\infty,0)\cup(L/\varepsilon^2,\infty), \end{cases} \tag{2.3}$$

$$\rho(\mathbf{x},z) = \bar\rho \quad \text{for} \quad \mathbf{x}\in\mathcal{D}, \quad z\in(-\infty,\infty). \tag{2.4}$$

It is possible to take into account a randomly varying density; this complicates the algebra but leads to the same general system of transport equations for the moments of reflected and transmitted mode amplitudes. Such a generalization was carried out in the case of randomly layered media in [5, section 17.3]. Here ε is a small parameter and $\nu(\mathbf{x},z)$ is a zero-mean random process that describes the random medium fluctuations that are mixing in the z-direction. This weakly heterogeneous regime can be encountered, for instance, in underwater acoustics [4, 8].

2.1. Waveguide modes. In a homogeneous waveguide $\nu = 0$, the complex amplitude of a monochromatic wave $p(t, \mathbf{x}, z) = \hat{p}(\omega, \mathbf{x}, z)e^{-i\omega t}$ at frequency ω satisfies the time-harmonic form of the wave equation (Helmholtz equation):

$$\partial_z^2 \hat{p} + \Delta_\perp \hat{p} + k^2(\omega)\hat{p} = 0. \tag{2.5}$$

Here $\Delta_\perp$ is the transverse Laplacian, $k(\omega) = \omega/\bar{c}$ is the wavenumber, and $\bar{c} = \sqrt{\bar{K}/\bar{\rho}}$ is the homogenized wave speed. The monochromatic wave can be decomposed in terms of normal modes which are the (normalized in $L^2(\mathcal{D})$) solutions of the eigenvalue problem

$$-\Delta_\perp \phi_j(\mathbf{x}) = \lambda_j \phi_j(\mathbf{x}),\ \mathbf{x} \in \mathcal{D}, \qquad \phi_j(\mathbf{x}) = 0,\ \mathbf{x} \in \partial\mathcal{D},$$

for $j = 1, 2, \dots$. The eigenvalues are positive and nondecreasing, and we assume for simplicity that they are simple, so we have $0 < \lambda_1 < \lambda_2 < \cdots$. The eigenmodes are real and form an orthonormal set

$$\int_{\mathcal{D}} \phi_j(\mathbf{x})\phi_l(\mathbf{x})\, d\mathbf{x} = \delta_{jl}.$$

For a given frequency ω, there exists a unique integer $N(\omega)$ such that $\lambda_{N(\omega)} \leq k^2(\omega) < \lambda_{N(\omega)+1}$, with the convention that $N(\omega) = 0$ if $\lambda_1 > k^2(\omega)$. The modal wavenumbers $\beta_j(\omega)$ for $1 \leq j \leq N(\omega)$ are defined by

$$\beta_j(\omega) = \sqrt{k^2(\omega) - \lambda_j}\,. \tag{2.6}$$

The solutions $\hat{p}_j(\omega, \mathbf{x}, z) = \phi_j(\mathbf{x})e^{\pm i\beta_j(\omega)z}$, $j = 1, \dots, N(\omega)$, of the wave equation (2.5) are the propagating waveguide modes. For $j > N(\omega)$ we define the modal wavenumbers by $\beta_j(\omega) = [\lambda_j - k^2(\omega)]^{1/2}$, and the corresponding solutions $\hat{q}_j(\omega, \mathbf{x}, z) = \phi_j(\mathbf{x})e^{\pm\beta_j(\omega)z}$ of the wave equation (2.5) are the evanescent modes.

From now on we consider the perturbed waveguide as described by (2.3)–(2.4). We expand the time-harmonic field inside the randomly perturbed waveguide in terms of the transverse eigenmodes of the unperturbed waveguide:

$$\hat{p}(\omega, \mathbf{x}, z) = \sum_{j=1}^{N(\omega)} \phi_j(\mathbf{x})\hat{p}_j(\omega, z) + \sum_{j=N(\omega)+1}^{\infty} \phi_j(\mathbf{x})\hat{q}_j(\omega, z), \tag{2.7}$$

where $\hat{p}_j$ is the amplitude of the jth propagating mode and $\hat{q}_j$ is the amplitude of the jth evanescent mode. For $1 \leq j \leq N(\omega)$, let $\hat{a}_j(\omega, z)$ and $\hat{b}_j(\omega, z)$ represent the amplitudes of the forward- and backward-propagating modes, with the forward direction referring to the z-direction. They are given by

$$\hat{p}_j(\omega, z) = \frac{1}{\sqrt{\beta_j(\omega)}} \left(\hat{a}_j(\omega, z)e^{i\beta_j(\omega)z} + \hat{b}_j(\omega, z)e^{-i\beta_j(\omega)z} \right), \tag{2.8}$$

$$\frac{d\hat{p}_j(\omega, z)}{dz} = i\sqrt{\beta_j(\omega)} \left(\hat{a}_j(\omega, z)e^{i\beta_j(\omega)z} - \hat{b}_j(\omega, z)e^{-i\beta_j(\omega)z} \right). \tag{2.9}$$

We next make a change of the z variable by introducing the rescaled processes $\hat{a}_j^\varepsilon(\omega, z)$, $\hat{b}_j^\varepsilon(\omega, z)$, $j = 1, \dots, N(\omega)$, given by

$$\hat{a}_j^\varepsilon(\omega, z) = \hat{a}_j\left(\omega, \frac{z}{\varepsilon^2}\right), \qquad \hat{b}_j^\varepsilon(\omega, z) = \hat{b}_j\left(\omega, \frac{z}{\varepsilon^2}\right). \tag{2.10}$$

By projecting the wave equation (2.5) on the transverse eigenmodes and by expressing the amplitudes of the evanescent modes in terms of the amplitudes of the propagating modes [5, 6], we obtain the following mode coupling equations for the amplitude processes $\hat{a}^\varepsilon(\omega,z) = (\hat{a}^\varepsilon_j(\omega,z))_{j=1,\dots,N(\omega)}$ and $\hat{b}^\varepsilon(\omega,z) = (\hat{b}^\varepsilon_j(\omega,z))_{j=1,\dots,N(\omega)}$:

$$\frac{d\hat{a}^\varepsilon}{dz} = \left[\frac{1}{\varepsilon}\mathbf{H}^{(aa)} + \mathbf{G}^{(aa)}\right]\left(\omega, \frac{z}{\varepsilon^2}\right)\hat{a}^\varepsilon + \left[\frac{1}{\varepsilon}\mathbf{H}^{(ab)} + \mathbf{G}^{(ab)}\right]\left(\omega, \frac{z}{\varepsilon^2}\right)\hat{b}^\varepsilon, \tag{2.11}$$

$$\frac{d\hat{b}^\varepsilon}{dz} = \left[\frac{1}{\varepsilon}\overline{\mathbf{H}^{(ab)}} + \overline{\mathbf{G}^{(ab)}}\right]\left(\omega, \frac{z}{\varepsilon^2}\right)\hat{a}^\varepsilon + \left[\frac{1}{\varepsilon}\overline{\mathbf{H}^{(aa)}} + \overline{\mathbf{G}^{(aa)}}\right]\left(\omega, \frac{z}{\varepsilon^2}\right)\hat{b}^\varepsilon, \tag{2.12}$$

with the two-point boundary conditions

$$\hat{a}^\varepsilon_j(\omega,0) = 0, \quad \hat{b}^\varepsilon_j(\omega,L) = \hat{b}^{\mathrm{inc}}_j(\omega), \tag{2.13}$$

which correspond to a left-propagating wave incoming from the right homogeneous waveguide. The matrices $\mathbf{G}$ describe coupling via evanescent modes [5]. The $N(\omega) \times N(\omega)$ coupling matrices have entries of form

$$H^{(aa)}_{jl}(\omega,z) = \frac{ik^2(\omega)}{2}\frac{C_{jl}(z)}{\sqrt{\beta_j\beta_l(\omega)}}e^{i(\beta_l(\omega)-\beta_j(\omega))z}, \tag{2.14}$$

$$G^{(aa)}_{jl}(\omega,z) = \frac{ik^4(\omega)}{4}\sum_{l'>N(\omega)}\int_{-\infty}^{\infty}\frac{C_{jl'}(z)C_{ll'}(z+s)}{\sqrt{\beta_j\beta_{l'}^2\beta_l(\omega)}}e^{i\beta_l(\omega)(z+s)-i\beta_j(\omega)z-\beta_{l'}(\omega)|s|}\,ds,$$

$$H^{(ab)}_{jl}(\omega,z) = -e^{-2i\beta_j(\omega)z}\overline{H^{(aa)}_{jl}(\omega,z)}, \quad G^{(ab)}_{jl}(\omega,z) = -e^{-2i\beta_j(\omega)z}\overline{G^{(aa)}_{jl}(\omega,z)}, \tag{2.15}$$

$$C_{jl}(z) = \int_{\mathcal{D}}\phi_j(\mathbf{x})\phi_l(\mathbf{x})\nu(\mathbf{x},z)\,d\mathbf{x} \tag{2.16}$$

for $j,l = 1,\dots,N(\omega)$.

2.2. Channel coupled wave approximation. We use an invariant imbedding step to convert the boundary value problem to an initial value problem with the objective being to characterize the reflected and transmitted wave fields. Accordingly we introduce the $N(\omega)\times N(\omega)$ reflection and transmission matrices $\boldsymbol{\mathcal{R}}^\varepsilon$ and $\boldsymbol{\mathcal{T}}^\varepsilon$ by

$$\hat{b}^\varepsilon(\omega,0) = \boldsymbol{\mathcal{T}}^\varepsilon(\omega,z)\hat{b}^\varepsilon(\omega,z), \qquad \hat{a}^\varepsilon(\omega,z) = \boldsymbol{\mathcal{R}}^\varepsilon(\omega,z)\hat{b}^\varepsilon(\omega,z). \tag{2.17}$$

Using (2.11)–(2.12) we find that these matrices solve the problems

$$\frac{d}{dz}\boldsymbol{\mathcal{R}}^\varepsilon = \mathbf{H}^{b,\varepsilon} + \mathbf{H}^{a,\varepsilon}\boldsymbol{\mathcal{R}}^\varepsilon - \boldsymbol{\mathcal{R}}^\varepsilon\overline{\mathbf{H}^{a,\varepsilon}} - \boldsymbol{\mathcal{R}}^\varepsilon\overline{\mathbf{H}^{b,\varepsilon}}\boldsymbol{\mathcal{R}}^\varepsilon, \tag{2.18}$$

$$\frac{d}{dz}\boldsymbol{\mathcal{T}}^\varepsilon = -\boldsymbol{\mathcal{T}}^\varepsilon\left(\overline{\mathbf{H}^{a,\varepsilon}} + \overline{\mathbf{H}^{b,\varepsilon}}\boldsymbol{\mathcal{R}}^\varepsilon\right), \tag{2.19}$$

where we defined

$$\mathbf{H}^{a,\varepsilon}(\omega,z) = \frac{1}{\varepsilon}\mathbf{H}^{(aa)}\left(\omega,\frac{z}{\varepsilon^2}\right) + \mathbf{G}^{(aa)}\left(\omega,\frac{z}{\varepsilon^2}\right),$$
$$\mathbf{H}^{b,\varepsilon}(\omega,z) = \frac{1}{\varepsilon}\mathbf{H}^{(ab)}\left(\omega,\frac{z}{\varepsilon^2}\right) + \mathbf{G}^{(ab)}\left(\omega,\frac{z}{\varepsilon^2}\right),$$

and where $\boldsymbol{\mathcal{R}}^\varepsilon(\omega,z)$ and $\boldsymbol{\mathcal{T}}^\varepsilon(\omega,z)$ take initial values at $z=0$

$$\boldsymbol{\mathcal{T}}^\varepsilon(\omega,0) = \mathbf{I}, \quad \boldsymbol{\mathcal{R}}^\varepsilon(\omega,0) = \mathbf{0}. \tag{2.20}$$

We remark that energy conservation leads to the reflection-transmission conservation relation

$$\boldsymbol{\mathcal{R}}^{\varepsilon\dagger}\boldsymbol{\mathcal{R}}^{\varepsilon}+\boldsymbol{\mathcal{T}}^{\varepsilon\dagger}\boldsymbol{\mathcal{T}}^{\varepsilon}=\mathbf{I}, \tag{2.21}$$

where the sign † stands for the conjugate transpose [5].

The initial value problem (2.18) is a stochastic Riccati matrix equation, and it can be analyzed in the limit $\varepsilon\to 0$ using the theory of diffusion approximations [5, 10]. The matrix $\mathbf{H}^{(aa)}$ contains rapidly varying phase factors and is centered with respect to the randomness that fluctuates on the scale ε^2 and is mixing in the z-direction. In this white-noise scaling regime we can then identify the corresponding infinitesimal generator and the associated white-noise model that describes the joint law of the transmission and reflection matrices in the limit $\varepsilon\to 0$.

3. The reflected wave field.

We consider the problem of characterizing the modal distribution of the reflected or transmitted waves. We consider in this section the reflected waves, and we will address the transmitted waves in section 4.

3.1. The moments of the reflected time-harmonic field.

We first consider the time-harmonic reflected field for a single frequency ω. We can compute the limit of the moments of the reflection matrix by using the diffusion approximation theory.

PROPOSITION 3.1. *Let* $\mathbf{p}=\{(j_1,l_1),\dots,(j_{|\mathbf{p}|},l_{|\mathbf{p}|})\}\in\{1,\dots,N(\omega)\}^{2|\mathbf{p}|}$ *denote a multi-index (*$|\mathbf{p}|$ *is the number of index pairs in* $\mathbf{p}$*). We introduce the moments of elements* $\mathcal{R}^{\varepsilon}_{jl}$ *of the reflection matrix:*

$$\mathcal{M}^{\varepsilon}_{\mathbf{p},\mathbf{q}}(\omega,z)=\mathbb{E}\Bigg[\prod_{(j,l)\in\mathbf{p}}\mathcal{R}^{\varepsilon}_{jl}(\omega,z)\prod_{(m,n)\in\mathbf{q}}\overline{\mathcal{R}^{\varepsilon}_{mn}(\omega,z)}\Bigg].$$

These moments converge as $\varepsilon\to 0$ *to the solution* $\mathcal{M}_{\mathbf{p},\mathbf{q}}$ *of the system*

$$\frac{d\mathcal{M}_{\mathbf{p},\mathbf{q}}}{dz}=D_{\mathbf{p},\mathbf{q}}(\omega)\mathcal{M}_{\mathbf{p},\mathbf{q}}+\big[\mathcal{S}_\omega(\mathcal{M})\big]_{\mathbf{p},\mathbf{q}}, \tag{3.1}$$

with the initial conditions

$$\mathcal{M}_{\mathbf{p},\mathbf{q}}(\omega,z=0)=\mathbf{1}_0(|\mathbf{p}|)\mathbf{1}_0(|\mathbf{q}|). \tag{3.2}$$

Here we have defined the linear operator $\mathcal{S}_\omega$*,*

$$\begin{aligned}\big[\mathcal{S}_\omega(\mathcal{M})\big]_{\mathbf{p},\mathbf{q}}&=-\sum_{(j,l)\in\mathbf{p}}d^{(1)}_{jl}\mathcal{M}_{\mathbf{p}|\{(j,l)|(l,j)\},\mathbf{q}}-\sum_{(j,l)\in\mathbf{q}}d^{(1)}_{jl}\mathcal{M}_{\mathbf{p},\mathbf{q}|\{(j,l)|(l,j)\}}\\
&-\sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{p}}d^{(2)}_{jl}\mathcal{M}_{\mathbf{p}|\{(j,l),(\tilde{j},\tilde{l})|(\tilde{j},j),(l,\tilde{l})\},\mathbf{q}}+d^{(2)}_{\tilde{j}\tilde{l}}\mathcal{M}_{\mathbf{p}|\{(j,l),(\tilde{j},\tilde{l})|(j,\tilde{j}),(\tilde{l},l)\},\mathbf{q}}\\
&-\sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{p}}d^{(1)}_{j\tilde{l}}\mathcal{M}_{\mathbf{p}|\{(j,l),(\tilde{j},\tilde{l})|(\tilde{j},j),(\tilde{l},l)\},\mathbf{q}}+d^{(1)}_{\tilde{j}l}\mathcal{M}_{\mathbf{p}|\{(j,l),(\tilde{j},\tilde{l})|(j,\tilde{j}),(l,\tilde{l})\},\mathbf{q}}\\
&-\sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{p}}\Big[d^{(5)}_{jl}+d^{(5)}_{\tilde{j}\tilde{l}}+d^{(1)}_{j\tilde{j}}+d^{(1)}_{l\tilde{l}}\Big]\mathcal{M}_{\mathbf{p}|\{(j,l),(\tilde{j},\tilde{l})|(\tilde{j},l),(j,\tilde{l})\},\mathbf{q}}\\
&-\sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{q}}d^{(2)}_{jl}\mathcal{M}_{\mathbf{p},\mathbf{q}|\{(j,l),(\tilde{j},\tilde{l})|(\tilde{j},j),(l,\tilde{l})\}}+d^{(2)}_{\tilde{j}\tilde{l}}\mathcal{M}_{\mathbf{p},\mathbf{q}|\{(j,l),(\tilde{j},\tilde{l})|(j,\tilde{j}),(\tilde{l},l)\}}\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{q}} d^{(1)}_{j\tilde{l}} \mathcal{M}_{\mathbf{p},\mathbf{q}|\{(j,l),(\tilde{j},\tilde{l})|(\tilde{j},j),(\tilde{l},l)\}} + d^{(1)}_{\tilde{j}l} \mathcal{M}_{\mathbf{p},\mathbf{q}|\{(j,l),(\tilde{j},\tilde{l})|(j,\tilde{j}),(l,\tilde{l})\}} \\
&- \sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{q}} \left[d^{(5)}_{jl} + d^{(5)}_{\tilde{j}\tilde{l}} + d^{(1)}_{j\tilde{j}} + d^{(1)}_{l\tilde{l}} \right] \mathcal{M}_{\mathbf{p},\mathbf{q}|\{(j,l),(\tilde{j},\tilde{l})|(\tilde{j},l),(j,\tilde{l})\}} \\
&+ \sum_{(j,l)\in\mathbf{p}} \sum_{(\tilde{j},\tilde{l})\in\mathbf{q}} d^{(3)}_{jl\tilde{j}\tilde{l}} \mathcal{M}_{\mathbf{p}|(j,l),\mathbf{q}|(\tilde{j},\tilde{l})} \\
&+ \sum_{(j,l)\in\mathbf{p}} \sum_{(\tilde{j},\tilde{l})\in\mathbf{q}} \sum_{k=1\neq j}^{N} \left[d^{(4)}_{jk\tilde{j}} \mathcal{M}_{\mathbf{p}|\{(j,l)|(k,l)\},\mathbf{q}|\{(\tilde{j},\tilde{l})|(k,\tilde{l})\}} + d^{(4)}_{jk\tilde{l}} \mathcal{M}_{\mathbf{p}|\{(j,l)|(k,l)\},\mathbf{q}|\{(\tilde{j},\tilde{l})|(\tilde{j},k)\}} \right] \\
&+ \sum_{(j,l)\in\mathbf{p}} \sum_{(\tilde{j},\tilde{l})\in\mathbf{q}} \sum_{k=1\neq l}^{N} \left[d^{(4)}_{lk\tilde{l}} \mathcal{M}_{\mathbf{p}|\{(j,l)|(j,k)\},\mathbf{q}|\{(\tilde{j},\tilde{l})|(\tilde{j},k)\}} + d^{(4)}_{lk\tilde{j}} \mathcal{M}_{\mathbf{p}|\{(j,l)|(j,k)\},\mathbf{q}|\{(\tilde{j},\tilde{l})|(k,\tilde{l})\}} \right] \\
&+ \sum_{(j,l)\in\mathbf{p}} \sum_{(\tilde{j},\tilde{l})\in\mathbf{q}} \sum_{k_1,k_2=1}^{N} d^{(2)}_{k_1k_2} \mathcal{M}_{\mathbf{p}|\{(j,l)|(j,k_1),(k_2,l)\},\mathbf{q}|\{(\tilde{j},\tilde{l})|(\tilde{j},k_1),(k_2,\tilde{l})\}} \\
&+ \sum_{(j,l)\in\mathbf{p}} \sum_{(\tilde{j},\tilde{l})\in\mathbf{q}} \sum_{k_1,k_2=1}^{N} d^{(5)}_{k_1k_2} \mathcal{M}_{\mathbf{p}|\{(j,l)|(j,k_1),(k_2,l)\},\mathbf{q}|\{(\tilde{j},\tilde{l})|(\tilde{j},k_2),(k_1,\tilde{l})\}},
\end{aligned}
$$

and we have used the following notation: If $\mathbf{p}$ *is a multi-index and* $\{(j_1,l_1),\ldots,(j_m,l_m)\} \subset \mathbf{p}$*, then* $\mathbf{p}|\{(j_1,l_1),\ldots,(j_m,l_m)|(\tilde{j}_1,\tilde{l}_1),\ldots,(\tilde{j}_n,\tilde{l}_n)\}$ *denotes the new multi-index obtained from* $\mathbf{p}$ *by removing the index pairs* $\{(j_1,l_1),\ldots,(j_m,l_m)\}$ *and by adding the new index pairs* $\{(\tilde{j}_1,\tilde{l}_1),\ldots,(\tilde{j}_n,\tilde{l}_n)\}$*. Finally, the coefficients* $D_{\mathbf{p},\mathbf{q}}(\omega)$ *and* $d^{(j)}(\omega)$*,* $j=1,\ldots,5$*, are defined by*

$$
\begin{aligned}
D_{\mathbf{p},\mathbf{q}} = {} & i \sum_{(j,l)\in\mathbf{p}} \big(\kappa_j + \kappa_l\big) - i \sum_{(j,l)\in\mathbf{q}} \big(\kappa_j + \kappa_l\big) \\
& - \sum_{(j,l)\in\mathbf{p}} \sum_{k=1}^{N} \Big(\overline{\Gamma_{jk}} + \overline{\Gamma_{lk}} + \widetilde{\Gamma}_{jk} + \widetilde{\Gamma}_{lk} \Big) - \sum_{(j,l)\in\mathbf{q}} \sum_{k=1}^{N} \Big(\Gamma_{jk} + \Gamma_{lk} + \overline{\widetilde{\Gamma}_{jk}} + \overline{\widetilde{\Gamma}_{lk}} \Big) \\
& - 2 \sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{p}} \Big(\check{\Gamma}_{j\tilde{j}} + \check{\Gamma}_{l\tilde{l}} \Big) - 2 \sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{q}} \Big(\check{\Gamma}_{j\tilde{j}} + \check{\Gamma}_{l\tilde{l}} \Big) \\
& - \sum_{(j,l)\in\mathbf{p}} \sum_{(\tilde{j},\tilde{l})\in\mathbf{p}} 2\Re\big(\check{\Gamma}_{j\tilde{l}}\big) - \sum_{(j,l)\in\mathbf{q}} \sum_{(\tilde{j},\tilde{l})\in\mathbf{q}} 2\Re\big(\check{\Gamma}_{j\tilde{l}}\big) \\
& + 2 \sum_{(j,l)\in\mathbf{p}} \sum_{(\tilde{j},\tilde{l})\in\mathbf{q}} \Big(\check{\Gamma}_{j\tilde{l}} + \check{\Gamma}_{l\tilde{j}} + \check{\Gamma}_{\tilde{j}j} + \check{\Gamma}_{\tilde{l}l} \Big),
\end{aligned}
$$

$$
\begin{aligned}
& d^{(1)}_{jl}(\omega) = 2\Re\big[\widetilde{\Gamma}_{jl}(\omega)\big]\mathbf{1}_{j\neq l}, \qquad d^{(2)}_{jl}(\omega) = 2\Re\big[\Gamma_{jl}(\omega)\big], \\
& d^{(3)}_{jl\tilde{j}\tilde{l}}(\omega) = 2\Re\big[\Gamma_{jl}(\omega)\big]\mathbf{1}_{(j,l)=(\tilde{j},\tilde{l}) \ or \ (j,l)=(\tilde{l},\tilde{j})}, \\
& d^{(4)}_{jk\tilde{j}}(\omega) = 2\Re\big[\widetilde{\Gamma}_{jk}(\omega)\big]\mathbf{1}_{j=\tilde{j}}, \qquad d^{(5)}_{k_1k_2}(\omega) = 2\Re\big[\Gamma_{k_1k_2}(\omega)\big]\mathbf{1}_{k_1\neq k_2},
\end{aligned}
$$

with $\Re(x)$ *the real part of* x*,* $\mathbf{1}_{j\neq l} = 1$ *if* $j \neq l$ *and* 0 *otherwise, and*

$$
\check{\Gamma}_{jl}(\omega) = \frac{k^4(\omega)}{4} \frac{\int_0^\infty \mathbb{E}[C_{jj}(0)C_{ll}(s)]\,ds}{\beta_j\beta_l(\omega)}, \tag{3.3}
$$

$$\widetilde{\Gamma}_{jl}(\omega) = \frac{k^4(\omega)}{4} \frac{\int_0^\infty e^{i(\beta_j(\omega)-\beta_l(\omega))s}\mathbb{E}[C_{jl}(0)C_{jl}(s)]\, ds}{\beta_j\beta_l(\omega)}, \tag{3.4}$$

$$\Gamma_{jl}(\omega) = \frac{k^4(\omega)}{4} \frac{\int_0^\infty e^{i(\beta_j(\omega)+\beta_l(\omega))s}\mathbb{E}[C_{jl}(0)C_{jl}(s)]\, ds}{\beta_j\beta_l(\omega)}, \tag{3.5}$$

$$\kappa_l(\omega) = \frac{k^4(\omega)}{4} \sum_{l'>N(\omega)} \frac{\int_{-\infty}^{\infty} \mathbb{E}\left[C_{ll'}(0)\, C_{ll'}(s)\right] e^{i\beta_l(\omega)s-\beta_{l'}(\omega)|s|}\, ds}{\beta_l\beta_{l'}(\omega)}. \tag{3.6}$$

We will discuss applications of this proposition in the next sections, but we first give a generalization of the result for the two-frequency case. The proof of Proposition 3.1 is a simplified version of the proof of the next proposition, so we shall present it only for this second proposition.

3.2. The transport equations for the two-frequency moments. We have the following result, which is proved in Appendix A. We use the same notation as in Proposition 3.1.

PROPOSITION 3.2. *We introduce the moments of elements $\mathcal{R}^\varepsilon_{jl}$ of the reflection matrix at two nearby frequencies:*

$$\mathcal{U}^\varepsilon_{\mathbf{p},\mathbf{q}}(\omega,h,z) = \mathbb{E}\Bigg[\prod_{(j,l)\in\mathbf{p}} \mathcal{R}^\varepsilon_{jl}(\omega+\varepsilon^2 h/2,z) \prod_{(m,n)\in\mathbf{q}} \overline{\mathcal{R}^\varepsilon_{mn}(\omega-\varepsilon^2 h/2,z)}\Bigg], \tag{3.7}$$

where we set $\mathcal{R}^\varepsilon_{jl}(\omega\pm\varepsilon^2 h/2,z)=0$ if j or l is larger than $N(\omega\pm\varepsilon^2 h/2)$. The family of Fourier transforms (in h)

$$\mathcal{W}^\varepsilon_{\mathbf{p},\mathbf{q}}(\omega,\tau,z) = \frac{1}{2\pi}\int e^{-ih[\tau-\phi_{\mathbf{p},\mathbf{q}}(\omega)z]}\mathcal{U}^\varepsilon_{\mathbf{p},\mathbf{q}}(\omega,h,z)\, dh \tag{3.8}$$

converges as $\varepsilon\to 0$ to the solution $\mathcal{W}_{\mathbf{p},\mathbf{q}}$ of the system of transport equations

$$\frac{\partial \mathcal{W}_{\mathbf{p},\mathbf{q}}}{\partial z} + \phi_{\mathbf{p},\mathbf{q}}(\omega)\frac{\partial \mathcal{W}_{\mathbf{p},\mathbf{q}}}{\partial \tau} = D_{\mathbf{p},\mathbf{q}}(\omega)\mathcal{W}_{\mathbf{p},\mathbf{q}} + \left[\mathcal{S}_\omega(\mathcal{W})\right]_{\mathbf{p},\mathbf{q}}, \tag{3.9}$$

with the initial conditions

$$\mathcal{W}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z=0) = \mathbf{1}_0(|\mathbf{p}|)\mathbf{1}_0(|\mathbf{q}|)\delta(\tau). \tag{3.10}$$

The coefficient $\phi_{\mathbf{p},\mathbf{q}}(\omega)$ is defined by

$$\phi_{\mathbf{p},\mathbf{q}}(\omega) = \frac{1}{2}\sum_{(j,l)\in\mathbf{p}}\left(\beta'_j(\omega)+\beta'_l(\omega)\right) + \frac{1}{2}\sum_{(j,l)\in\mathbf{q}}\left(\beta'_j(\omega)+\beta'_l(\omega)\right), \tag{3.11}$$

with $\beta'_j(\omega)=d\beta_j(\omega)/d\omega$, while the coefficient $D_{\mathbf{p},\mathbf{q}}(\omega)$ and the operator $\mathcal{S}_\omega$ are given in Proposition 3.1.

The set of transport equations (3.9) describes accurately the reflected wave field, and it is the key tool in analyzing various applications with waves in random waveguides. The corresponding transport equations in the layered case with one-dimensional medium variations were first obtained in [1]. They have played a crucial role in the analysis of a wide range of applications, and they have been generalized to describe a wide range of propagation scenarios in [5]. The transport equations given in Proposition 3.2 provide a rigorous tool for studying qualitatively and quantitatively the multiple scattering effects in a nonlayered random medium.

Remark. The convergence of $\mathcal{W}^\varepsilon$ and the existence and uniqueness of the solution $\mathcal{W}$ to the system of transport equations (3.9) are established in the space $\mathcal{C}([0,L],S'_H)$, where S'_H is a generalization of the space of distributions introduced in [12] to study the analogous problem with $N=1$ (randomly layered media). The space S'_H can be identified as the dual of the space S_H of the test functions $\lambda=(\lambda_{\mathbf{p},\mathbf{q}}(\tau))_{\mathbf{p}\in\{1,\dots,N(\omega)\}^{2|\mathbf{p}|},\mathbf{q}\in\{1,\dots,N(\omega)\}^{2|\mathbf{q}|},\tau\in\mathbb{R}}$, where the $\lambda_{\mathbf{p},\mathbf{q}}(\tau)$ are infinitely differentiable in τ and are rapidly decaying as functions of τ, $|\mathbf{p}|$ and $|\mathbf{q}|$. The convergence of $\mathcal{M}^\varepsilon$ in Proposition 3.1 is established in the space $\mathcal{C}([0,L],S'_M)$, where S'_M is the dual of the space S_M of the test sequences $\lambda=(\lambda_{\mathbf{p},\mathbf{q}})_{\mathbf{p}\in\{1,\dots,N(\omega)\}^{2|\mathbf{p}|},\mathbf{q}\in\{1,\dots,N(\omega)\}^{2|\mathbf{q}|}}$ which are rapidly decaying in $|\mathbf{p}|$ and $|\mathbf{q}|$.

3.3. Interpretation of the transport equations. We make the five following observations regarding the system of transport equations.

(1) By integrating the solution of the system of transport equations in τ, it is straightforward to see that the integral quantity is the solution of the system (3.1). This shows that we have

$$\mathcal{M}_{\mathbf{p},\mathbf{q}}(\omega,z)=\int \mathcal{W}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z)\,d\tau. \tag{3.12}$$

Therefore, the following remarks stated in terms of the family $\mathcal{W}_{\mathbf{p},\mathbf{q}}$ hold true for the family of moments $\mathcal{M}_{\mathbf{p},\mathbf{q}}$ as well.

(2) Consider the set of moments $\mathcal{W}_{\mathbf{p},\mathbf{q}}$ such that $|\mathbf{p}|-|\mathbf{q}|=c$ with c a nonzero integer. These moments form a closed subfamily with each member satisfying a zero initial condition. Therefore, these moments vanish and only moments having the same number of conjugated and unconjugated terms $|\mathbf{p}|=|\mathbf{q}|$ survive in the limit $\varepsilon\to 0$.

(3) Consider the case when

$$C_{jl}(z)\equiv 0 \quad \text{for} \quad j\neq l. \tag{3.13}$$

This corresponds to the situation where modes with different modal wavenumbers are not coupled. This is the case particularly when the inhomogeneities of the waveguide do not have lateral variations $\nu(\mathbf{x},z)=\nu(z)$. It then follows that

$$\widetilde{\Gamma}_{jl}(\omega)=\widetilde{\Gamma}_j^{(0)}(\omega)\mathbf{1}_{j=l}, \qquad \widetilde{\Gamma}_j^{(0)}(\omega)=\frac{k^4(\omega)}{2\beta_j^2(\omega)}\int_{-\infty}^{\infty}\mathbb{E}[\nu(0)\nu(s)]\,ds, \tag{3.14}$$

$$\Gamma_{jl}(\omega)=\Gamma_j^{(0)}(\omega)\mathbf{1}_{j=l}, \qquad \Gamma_j^{(0)}(\omega)=\frac{k^4(\omega)}{4\beta_j^2(\omega)}\int_0^{\infty}\mathbb{E}[\nu(0)\nu(s)]e^{i2\beta_j(\omega)s}\,ds. \tag{3.15}$$

This simplification gives $d_{jl}^{(1)}=0$, $d_{jl}^{(2)}=2\Gamma_j^{(0)}\mathbf{1}_{j=l}$, $d_{jl\tilde{j}\tilde{l}}^{(3)}=2\Gamma_j^{(0)}\mathbf{1}_{j=l=\tilde{j}=\tilde{l}}$, $d_{j\tilde{j}k}^{(4)}=2\widetilde{\Gamma}_j^{(0)}\mathbf{1}_{j=\tilde{j}=k}$, and $d_{k_1k_2}^{(5)}=0$. The analysis of the system shows that the solution has the form

$$\mathcal{W}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z)=\begin{cases} W_{p_1}^{(1)}*\cdots*W_{p_N}^{(N)}(\omega,\tau,z) \text{ if } \mathbf{p}=\mathbf{q}=\{(1,1)^{p_1},\dots,(N,N)^{p_N}\},\\ 0 \text{ otherwise,}\end{cases}$$

where $*$ stands for the convolution in τ and for each $j=1,\dots,N$ the family $(W_p^{(j)})_{p\in\mathbb{N}}$ is the solution of the closed system of transport equations

$$\frac{\partial W_p^{(j)}}{\partial z}+2p\beta_j'(\omega)\frac{\partial W_p^{(j)}}{\partial \tau}=2p^2\Re\left[\Gamma_j^{(0)}(\omega)\right]\left(W_{p+1}^{(j)}+W_{p-1}^{(j)}-2W_p^{(j)}\right), \tag{3.16}$$

with the initial conditions $W_p^{(j)}(\omega,\tau,z=0)=\mathbf{1}_0(p)\delta(\tau)$. We therefore obtain that the backward and forward jth modes are uncoupled from the other modes, but their moments are coupled together according to the system that governs the propagation of one-dimensional waves in random media [1]. This is not qualitatively surprising, but this analysis shows that a sufficient criterion for this reduction is (3.13).

(4) If the two-point statistics of the process $\nu(\mathbf{x},z)$ are such that

$$\Gamma_{jl}(\omega)\equiv 0 \quad \text{for all} \quad j,l=1,\dots,N(\omega), \tag{3.17}$$

then $d^{(2)}=d^{(3)}=d^{(5)}=0$. Consequently there is coupling in the system of transport equations only for indices $(\mathbf{p},\mathbf{q})$ and $(\mathbf{p}',\mathbf{q}')$ such that $|\mathbf{p}|=|\mathbf{p}'|$ and $|\mathbf{q}|=|\mathbf{q}'|$. Since the initial conditions are zero for all nonempty indices $(\mathbf{p},\mathbf{q})$, the moments $\mathcal{W}_{\mathbf{p},\mathbf{q}}$ are zero as soon as $|\mathbf{p}|$ or $|\mathbf{q}|$ is positive. In other words, $\mathcal{R}^{\varepsilon}_{jl}\to 0$ for all $j,l=1,\dots,N$ in distribution as $\varepsilon\to 0$. This shows that the forward scattering approximation is valid as soon as the condition (3.17) is fulfilled. This approximation is frequently used in the literature; it consists in neglecting coupling between forward- and backward-propagating modes, while retaining the coupling between forward-going modes and the implicit coupling to the evanescent modes. Here we give the necessary and sufficient condition (3.17) for the validity of this approximation.

(5) In the full system (2.18) we do not have"reciprocity" in that in general $\mathcal{R}^{\varepsilon}_{jl}\neq\mathcal{R}^{\varepsilon}_{lj}$ because of the coupling with the evanescent modes modeled by the matrices $\mathbf{G}$. However, the following symmetry relation is satisfied:

$$\mathcal{W}_{\mathbf{p},\mathbf{q}}=\mathcal{W}_{\tilde{\mathbf{p}},\tilde{\mathbf{q}}}$$

for $\tilde{\mathbf{p}}_n=(l_n,j_n)$ with $\mathbf{p}_n=(j_n,l_n)$ and $\tilde{\mathbf{q}}$ correspondingly defined. This means that reciprocity is satisfied in the limit $\varepsilon\to 0$, and this follows from the following observations:

- The initial condition in (3.10) depends on the multi-index only through $|\mathbf{p}|$ and $|\mathbf{q}|$.
- The coupling matrices $\mathbf{G}^{(aa)}$ and $\mathbf{G}^{(ab)}$ in (2.18) affect only the diagonal coefficients $D_{\mathbf{p},\mathbf{q}}$ in a symmetric way in the problem for $\mathcal{W}_{\mathbf{p},\mathbf{q}}$.
- We have the symmetry relations

$$\left(\mathbf{H}^{(aa)}\right)^T=-\overline{\mathbf{H}^{(aa)}},\quad \left(\mathbf{H}^{(ab)}\right)^T=\mathbf{H}^{(ab)}$$

  in the coupling matrices in (2.18).

3.4. Enhanced backscattering. In this section we consider the case where the forward coupling is strong while the coupling between the forward- and backward-going modes is weak. As seen above, the forward scattering approximation consists in neglecting completely the latter coupling, and it is valid when the matrix Γ is zero. Here we assume that Γ is not zero but Γ is small compared to $\widetilde{\Gamma}$. This allows us to simplify significantly the system of transport equations and to present very interesting results. In particular, we show in the following proposition that the enhanced backscattering phenomenon extensively discussed in the physical literature can be exhibited from the particular structure of the reflected time-harmonic wave field.

We denote by $\mathcal{P}_{jl}$ the mean reflected power of the mode j when the input wave is a mode l:

$$\mathcal{P}^{(1)}_{jl}(\omega,z)=\mathcal{M}_{(j,l),(j,l)}(\omega,z)=\lim_{\varepsilon\to 0}\mathbb{E}[|\mathcal{R}^{\varepsilon}_{jl}(\omega,z)|^2],\qquad j,l=1,\dots,N(\omega),$$

for a random waveguide with length z.

PROPOSITION 3.3. *If the matrix norm of* $\Re[\Gamma(\omega)]$ *is small compared to* $1/L$ *and the positive spectral gap* (3.20) *of the operator* $\mathcal{L}_\omega$ *in* (3.18) *below, which is defined in terms of* $\widetilde{\Gamma}(\omega)$*, is large compared to* $1/L$*, then the mean reflected mode powers are*

$$\mathcal{P}^{(1)}_{jl}(\omega, L) = \begin{cases} P_0(\omega) & \text{if } j \neq l, \\ 2P_0(\omega) & \text{if } j = l, \end{cases}$$

where $P_0(\omega)$ *is given by*

$$P_0(\omega) = \frac{2}{N(\omega)(N(\omega)+3)} \left[\sum_{j\neq l} \Re[\Gamma_{jl}(\omega)] + 2 \sum_j \Re[\Gamma_{jj}(\omega)] \right] L.$$

The first condition "$\Re[\Gamma(\omega)L] \ll 1$" means that the coupling between backward- and forward-going modes is weak. The second condition about the spectral gap means that the coupling between forward-going modes is strong (as well as the coupling between backward-going modes). The most striking result of this proposition is that, if the incident wave is a pure mode l, then the mean reflected power of the lth mode $\mathcal{P}^{(1)}_{ll}$ is twice the mean reflected power of any other mode $\mathcal{P}^{(1)}_{jl}$, $j \neq l$.

First, this result shows that the reflected wave has a memory of the initial conditions. This is in contrast to the transmitted wave field in the same regime, where the equipartition of energy means that the mean transmitted mode powers acquire a uniform distribution over the modes, independently of the initial conditions (see section 4.4).

Second, since a mode corresponds to a particular wavevector angle, this result means that we observe a uniform mean reflected power in all outgoing directions, except in the backscattered direction (corresponding to the input one), where we observe twice as much power. The physical reason for this enhancement of backscattered power is the constructive interference between the direct and reverse paths in the backscattering direction. Enhanced backscattering was first predicted in three-dimensional random media in [2] and was detected by several groups [9, 15, 17]. It is also referred to as the weak localization effect. The most popular techniques amongst physicists for analyzing the weak localization effect, and more generally for taking into account interference effects, are based on diagrammatic expansions [16]. Here we give a mathematical derivation of this phenomenon in the context of random waveguides.

Proof. The initial conditions for the solution $\mathcal{W}_{\mathbf{p},\mathbf{q}}$ of the system of transport equations is zero as soon as $|\mathbf{p}| > 0$ or $|\mathbf{q}| > 0$. Since the coupling terms from $|\mathbf{p}|$ to $|\mathbf{p}| \pm 1$ and from $|\mathbf{q}|$ to $|\mathbf{q}| \pm 1$ are proportional to $\Re(\Gamma)$, this shows that the only coefficients $\mathcal{W}_{\mathbf{p},\mathbf{q}}$ of order $\Re(\Gamma)$ are the ones with $|\mathbf{p}| = 1$ and $|\mathbf{q}| = 1$. Up to terms of higher order, we find that

$$\mathcal{W}_{\mathbf{p},\mathbf{q}}(\omega, \tau, z) = \begin{cases} \delta(\tau) & \text{if } \mathbf{p} = \mathbf{q} = \emptyset, \\ W_{jl}(\omega, \tau, z) & \text{if } \{\mathbf{p} = (j,l),\, \mathbf{q} = (j,l)\} \text{ or } \{\mathbf{p} = (j,l),\, \mathbf{q} = (l,j)\}, \\ 0 & \text{otherwise}, \end{cases}$$

where W_{jl} is the solution of

$$\frac{\partial W_{jl}}{\partial z} + \left[\beta_j'(\omega) + \beta_l'(\omega)\right] \frac{\partial W_{jl}}{\partial \tau} = (\mathcal{L}_\omega W)_{jl} + 2\Re\left[\Gamma_{jl}(\omega)\right]\delta(\tau),$$

with the initial conditions $W_{jl}(\omega, \tau, z) = 0$. Here $\mathcal{L}_\omega$ is the linear operator from $\mathbb{R}^{N\times N}$ into $\mathbb{R}^{N\times N}$ defined by

$$(3.18)\qquad (\mathcal{L}_\omega \mathcal{P})_{jl} = \begin{cases} \sum_{k\neq j} \tilde{\gamma}_{jk}(\mathcal{P}_{kl} - \mathcal{P}_{jl}) + \sum_{k\neq l} \tilde{\gamma}_{kl}(\mathcal{P}_{jk} - \mathcal{P}_{jl}) - 2\tilde{\gamma}_{jl}\mathcal{P}_{jl} & \text{if } j \neq l, \\ 2\sum_{k\neq j} \tilde{\gamma}_{jk}(2\mathcal{P}_{jk} - \mathcal{P}_{jj}) & \text{if } j = l, \end{cases}$$

where $\tilde{\gamma}_{jl}(\omega) = 2\Re[\widetilde{\Gamma}_{jl}(\omega)]$. By integrating in τ and using (3.12), we obtain the system for the mean reflected powers

$$(3.19)\qquad \frac{d\mathcal{P}^{(1)}_{jl}}{dz} = (\mathcal{L}_\omega \mathcal{P}^{(1)})_{jl} + 2\Re\big[\Gamma_{jl}(\omega)\big].$$

Interpreting $\mathcal{L}_\omega$ as an $N^2(\omega) \times N^2(\omega)$ matrix acting on $N^2(\omega)$-dimensional vectors, it is straightforward to check that the vector $\mathcal{P}^*(\omega)$ defined by

$$\mathcal{P}^*_{jl}(\omega) = \begin{cases} \dfrac{1}{\sqrt{N(\omega)(N(\omega)+3)}} & \text{if } j \neq l, \\ \dfrac{2}{\sqrt{N(\omega)(N(\omega)+3)}} & \text{if } j = l \end{cases}$$

is a unit eigenvector of $\mathcal{L}_\omega$ associated with the eigenvalue zero. Additionally, using the positivity of the matrix $\Re(\widetilde{\Gamma})$ and the Perron–Frobenius theorem, one can show that zero is a simple eigenvalue and all other eigenvalues are negative. Let us denote by $\lambda^{(N^2)}(\omega) \leq \cdots \leq \lambda^{(2)}(\omega) < 0$ these eigenvalues and by $\mathcal{Q}^{(N^2)}(\omega), \ldots, \mathcal{Q}^{(2)}(\omega)$ the corresponding unit eigenvectors. The spectral gap mentioned in the proposition is $|\lambda^{(2)}(\omega)|$, which is also given by

$$(3.20)\qquad |\lambda^{(2)}(\omega)| = \inf_{\mathcal{P}\in\mathbb{R}^{N^2(\omega)},\, \langle \mathcal{P},\mathcal{P}^*(\omega)\rangle = 0} \frac{-\langle \mathcal{P}, \mathcal{L}_\omega \mathcal{P}\rangle}{\langle \mathcal{P}, \mathcal{P}\rangle},$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{N^2(\omega)}$. The integration of (3.19) gives

$$(3.21)\qquad \mathcal{P}^{(1)}_{jl}(\omega,z) = \mathcal{P}^*_{jl}\langle \mathcal{P}^*, 2\Re(\Gamma)\rangle z + \sum_{k=2}^{N^2} \mathcal{Q}^{(k)}_{jl} \left\langle \mathcal{Q}^{(k)}, 2\Re(\Gamma)\right\rangle \frac{\exp(\lambda^{(k)} z) - 1}{\lambda^{(k)}}.$$

If $|\lambda^{(2)}|z$ is much larger than 1, then the first term of the right-hand side is much larger than the other terms. This gives the desired result. ∎

3.5. Fluctuation theory for the reflected mode powers. The previous section describes the mean reflected powers. It is important to study the fluctuations of the reflected powers in order to predict under which conditions the enhanced backscattering can be observed. Propositions 3.1–3.2 allow us to study the fluctuations of the reflected mode powers by looking at their second moments:

$$\mathcal{P}^{(2)}_{jl,mn}(\omega,z) = \lim_{\varepsilon\to 0} \mathbb{E}\left[|\mathcal{R}^\varepsilon_{jl}(\omega,z)|^2 |\mathcal{R}^\varepsilon_{mn}(\omega,z)|^2\right].$$

We investigate the asymptotic correlation matrix (of size $N^2(\omega) \times N^2(\omega)$) of the reflected mode powers:

$$\begin{aligned} \mathrm{Cor}_{jl,mn}(\omega) &= \lim_{\varepsilon\to 0} \frac{\mathbb{E}\left[|\mathcal{R}^\varepsilon_{jl}(\omega,L)|^2 |\mathcal{R}^\varepsilon_{mn}(\omega,L)|^2\right] - \mathbb{E}\Big[|\mathcal{R}^\varepsilon_{jl}(\omega,L)|^2\Big]\mathbb{E}\Big[|\mathcal{R}^\varepsilon_{mn}(\omega,L)|^2\Big]}{\mathbb{E}\Big[|\mathcal{R}^\varepsilon_{jl}(\omega,L)|^2\Big]\mathbb{E}\Big[|\mathcal{R}^\varepsilon_{mn}(\omega,L)|^2\Big]} \\ &= \frac{\mathcal{P}^{(2)}_{jl,mn}(\omega,L) - \mathcal{P}^{(1)}_{jl}(\omega,L)\mathcal{P}^{(1)}_{mn}(\omega,L)}{\mathcal{P}^{(1)}_{jl}(\omega,L)\mathcal{P}^{(1)}_{mn}(\omega,L)}. \end{aligned}$$

PROPOSITION 3.4. *If the matrix norm of $\Re[(\Gamma(\omega)]$ is small compared to $1/L$ and the positive spectral gap of the operator $\mathcal{L}_\omega^{(2)}$ given in 1–7 below, which is defined in terms of $\widetilde{\Gamma}(\omega)$, is large compared to $1/L$, then the second moments of the reflected mode powers satisfy*

$$\lim_{N(\omega)\to\infty} \mathrm{Cor}_{jl,mn}(\omega) = \begin{cases} 0 \text{ if } (j,l) \neq (m,n) \text{ and } (j,l) \neq (n,m), \\ 1 \text{ if } (j,l) = (m,n) \text{ or } (j,l) = (n,m). \end{cases}$$

The result for $(j,l) \neq (m,n)$ shows that the reflected mode powers are asymptotically uncorrelated as $N \to \infty$. The result for $(j,l) = (m,n)$ shows that they are not statistically stable quantities as $N \to \infty$, since their normalized variances are equal to one. This means that the fluctuations of the reflected mode powers are of the same order as their mean values. This implies that it is necessary to perform an averaging in order to observe the enhanced backscattering. This averaging can be done by a summation of the reflected mode powers over different experiments with different realizations of the random medium, or with the same realization of the random medium but with different frequencies of the input monochromatic wave.

Another interesting point is that the normalized variances of the background reflected powers (i.e., $\mathrm{Cor}_{jl,jl}$ for $j \neq l$) are asymptotically equal to one and equal to the normalized variance of the backscattered reflected power (i.e., $\mathrm{Cor}_{jj,jj}$).

Proof. We apply the same strategy as in the proof of Proposition 3.3. Once again, the fundamental argument is that the coupling terms from $|\mathbf{p}|$ to $|\mathbf{p}| \pm 1$ are proportional to $\Re(\Gamma)$. Therefore, the lowest order terms in $\Re(\Gamma)$ of the coefficients $\mathcal{W}_{\mathbf{p},\mathbf{q}}$ with $|\mathbf{p}| = |\mathbf{q}| = 2$ are

$$\mathcal{W}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z) = \begin{cases} W_{jl,mn}(\omega,\tau,z) \text{ if } & \begin{array}{l} \mathbf{p} = \{(j,l),(m,n)\} \text{ and } \mathbf{q} = \{(j,l),(m,n)\} \\ \text{or } \{(l,j),(m,n)\} \text{ or } \{(j,l),(n,m)\} \\ \text{or } \{(l,j),(n,m)\}, \end{array} \\ 0 \text{ otherwise,} \end{cases}$$

where $W_{jl,mn}$ is the solution of

$$\frac{\partial W_{jl,mn}}{\partial z} + (\beta_j' + \beta_l' + \beta_m' + \beta_n') \frac{\partial W_{jl,mn}}{\partial \tau} = (\mathcal{L}_\omega^{(2)} W)_{jl,mn} + 2\Re(\Gamma_{jl}) W_{mn} + 2\Re(\Gamma_{mn}) W_{jl},$$

with the initial conditions $W_{jl,mn}(\omega,\tau,z=0) = 0$. Here $\mathcal{L}_\omega^{(2)}$ is the linear operator from $\mathbb{R}^{N^2\times N^2}$ into $\mathbb{R}^{N^2\times N^2}$ defined by the following:

1. If $j = l = m = n$,

$$(\mathcal{L}_\omega^{(2)} \mathcal{P}^{(2)})_{jj,jj} = \sum_{k\neq j} \tilde{\gamma}_{jk} [16\mathcal{P}^{(2)}_{jk,jj} - 4\mathcal{P}^{(2)}_{jj,jj}],$$

where $\tilde{\gamma}_{jl}(\omega) = 2\Re[\widetilde{\Gamma}_{jl}(\omega)]$.

2. If $j = l = m \neq n$,

$$(\mathcal{L}_\omega^{(2)} \mathcal{P}^{(2)})_{jj,jn} = \sum_{k\neq j} \tilde{\gamma}_{jk} [4\mathcal{P}^{(2)}_{jk,jn} - 2\mathcal{P}^{(2)}_{jj,jn}] + \sum_{k\neq j} \tilde{\gamma}_{jk} [\mathcal{P}^{(2)}_{jj,kn} - \mathcal{P}^{(2)}_{jj,jn}]$$
$$+ \sum_{k\neq m} \tilde{\gamma}_{nk} [\mathcal{P}^{(2)}_{jj,jk} - \mathcal{P}^{(2)}_{jj,jn}] - 6\tilde{\gamma}_{jn} \mathcal{P}^{(2)}_{jj,jn}.$$

A formula of the same form holds true if $j = l = n \neq m$ or $j \neq l = m = n$ or $l \neq j = m = n$.

3. If $j = l \neq m = n$,

$$(\mathcal{L}_\omega^{(2)}\mathcal{P}^{(2)})_{jj,nn} = \sum_{k\neq j} \tilde{\gamma}_{jk}[4\mathcal{P}^{(2)}_{jk,nn} - 2\mathcal{P}^{(2)}_{jj,nn}] + \sum_{k\neq n} \tilde{\gamma}_{nk}[4\mathcal{P}^{(2)}_{jj,kn} - 2\mathcal{P}^{(2)}_{jj,nn}].$$

4. If $j = m \neq l = n$,

$$(\mathcal{L}_\omega^{(2)}\mathcal{P}^{(2)})_{jl,jl} = \sum_{k\neq j} \tilde{\gamma}_{jk}[4\mathcal{P}^{(2)}_{kl,jl} - 2\mathcal{P}^{(2)}_{jl,jl}] + \sum_{k\neq l} \tilde{\gamma}_{lk}[4\mathcal{P}^{(2)}_{jk,jl} - 2\mathcal{P}^{(2)}_{jl,jl}] - 4\tilde{\gamma}_{jl}\mathcal{P}^{(2)}_{jl,jl}.$$

A formula of the same form holds true if $j = n \neq l = m$.

5. If $j = l \neq m \neq n$,

$$\begin{aligned}(\mathcal{L}_\omega^{(2)}\mathcal{P}^{(2)})_{jj,mn} = &\sum_{k\neq j} \tilde{\gamma}_{jk}[4\mathcal{P}^{(2)}_{jk,mn} - 2\mathcal{P}^{(2)}_{jj,mn}] + \sum_{k\neq m} \tilde{\gamma}_{mk}[\mathcal{P}^{(2)}_{jj,kn} - \mathcal{P}^{(2)}_{jj,mn}] \\ &+ \sum_{k\neq n} \tilde{\gamma}_{nk}[\mathcal{P}^{(2)}_{jj,mk} - \mathcal{P}^{(2)}_{jj,mn}] - 2\tilde{\gamma}_{mn}\mathcal{P}^{(2)}_{jj,mn}.\end{aligned}$$

A formula of the same form holds true if $m = n \neq j \neq l$.

6. If $j = m \neq l \neq n$,

$$\begin{aligned}(\mathcal{L}_\omega^{(2)}\mathcal{P}^{(2)})_{jl,jn} = &\sum_{k\neq j} \tilde{\gamma}_{jk}[\mathcal{P}^{(2)}_{kl,jn} - \mathcal{P}^{(2)}_{jl,jn}] + \sum_{k\neq l} \tilde{\gamma}_{lk}[\mathcal{P}^{(2)}_{jk,jn} - \mathcal{P}^{(2)}_{jl,jn}] \\ &+ \sum_{k\neq j} \tilde{\gamma}_{jk}[\mathcal{P}^{(2)}_{jl,kn} - \mathcal{P}^{(2)}_{jl,jn}] + \sum_{k\neq n} \tilde{\gamma}_{nk}[\mathcal{P}^{(2)}_{jl,jk} - \mathcal{P}^{(2)}_{jl,jn}] \\ &- 2\,[\tilde{\gamma}_{jl} + \tilde{\gamma}_{jn} + \tilde{\gamma}_{ln}]\,\mathcal{P}^{(2)}_{jl,jn}.\end{aligned}$$

A formula of the same form holds true if $j = n \neq l \neq m$ or $l = m \neq j \neq n$ or $l = n \neq j \neq m$.

7. In the other cases,

$$\begin{aligned}(\mathcal{L}_\omega^{(2)}\mathcal{P}^{(2)})_{jl,mn} = &\sum_{k\neq j} \tilde{\gamma}_{jk}[\mathcal{P}^{(2)}_{kl,mn} - \mathcal{P}^{(2)}_{jl,mn}] + \sum_{k\neq l} \tilde{\gamma}_{lk}[\mathcal{P}^{(2)}_{jk,mn} - \mathcal{P}^{(2)}_{jl,mn}] \\ &+ \sum_{k\neq m} \tilde{\gamma}_{mk}[\mathcal{P}^{(2)}_{jl,kn} - \mathcal{P}^{(2)}_{jl,mn}] + \sum_{k\neq n} \tilde{\gamma}_{nk}[\mathcal{P}^{(2)}_{jl,mk} - \mathcal{P}^{(2)}_{jl,mn}] \\ &- 2\,[\tilde{\gamma}_{jl} + \tilde{\gamma}_{mn}]\,\mathcal{P}^{(2)}_{jl,mn}.\end{aligned}$$

By integrating in τ, we find the system for the second moments of the reflected powers:

$$\frac{d\mathcal{P}^{(2)}_{jl,mn}}{dz} = (\mathcal{L}_\omega^{(2)}\mathcal{P}^{(2)})_{jl,mn} + 2\Re(\Gamma_{jl})\mathcal{P}^{(1)}_{mn} + 2\Re(\Gamma_{mn})\mathcal{P}^{(1)}_{jl}. \tag{3.22}$$

Interpreting $\mathcal{L}_\omega^{(2)}$ as an $N^4(\omega)\times N^4(\omega)$ matrix acting on $N^4(\omega)$-dimensional vectors, it is possible to check that the vector $\mathcal{P}^{(2),*}(\omega)$ defined by

$$\mathcal{P}^{(2),*}_{jl,mn} = \frac{1}{\sqrt{T^*}}\begin{cases} 8 & \text{if } j = l = m = n, \\ 4 & \text{if } j = l \neq m = n, \\ 2 & \text{if } \left|\begin{array}{l} j = m \neq l = n \text{ or } j = n \neq l = m \text{ or } j = l \neq m \neq n \\ \text{or } j \neq l \neq m = n \text{ or } j = l = m \neq n \text{ or } j = l = n \neq m \\ \text{or } j \neq l = m = n \text{ or } l \neq j = m = n, \end{array}\right. \\ 1 & \text{otherwise} \end{cases}$$

is the unit eigenvector of $\mathcal{L}_\omega^{(2)}$ associated with the eigenvalue zero, where $T^* = N^4+6N^3+15N^2+42N$. The other eigenvalues are negative. As a result, taking into account the expression in (3.21) for $\mathcal{P}^{(1)}$, the integration of (3.22) gives

$$\frac{1}{z^2}\mathcal{P}^{(2)}_{jl,mn}(\omega,z) \xrightarrow{z\to\infty} \mathcal{P}^{(2),*}_{jl,mn} \sum_{\tilde{j},\tilde{l},\tilde{m},\tilde{n}=1}^{N} \mathcal{P}^{(2),*}_{\tilde{j}\tilde{l},\tilde{m}\tilde{n}} \left(\Re(\Gamma_{\tilde{j}\tilde{l}})\mathcal{P}^*_{\tilde{m}\tilde{n}} + \Re(\Gamma_{\tilde{m}\tilde{n}})\mathcal{P}^*_{\tilde{j}\tilde{l}}\right)\langle \mathcal{P}^*, 2\Re(\Gamma)\rangle\,,$$

which in turn gives the result of the proposition. $\square$

4. The transmitted wave field. We consider the problem of characterizing the distribution of the transmitted field.

4.1. The moments of the transmitted time-harmonic field. We consider the time-harmonic transmitted field for a frequency ω. We can compute the limit of moments of the transmission matrix as in the case of the reflection matrix. We use the same notation as in Proposition 3.1.

PROPOSITION 4.1. *We introduce the joint moments of elements of the reflection matrix along with a pair of elements of the transmission matrix:*

$$\mathcal{M}^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}(\omega,z;j_1,j_2) = \mathbb{E}\left[\mathcal{T}^\varepsilon_{j_1l_1}(\omega,z)\overline{\mathcal{T}^\varepsilon_{j_2l_2}(\omega,z)}\prod_{(j,l)\in\mathbf{p}}\mathcal{R}^\varepsilon_{jl}(\omega,z)\prod_{(m,n)\in\mathbf{q}}\overline{\mathcal{R}^\varepsilon_{mn}(\omega,z)}\right] \tag{4.1}$$

for $\mathbf{t}=(l_1,l_2)$. *The family of moments* $\mathcal{M}^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}$ *converges as* $\varepsilon\to 0$ *to the solution* $\mathcal{M}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}$ *of the system*

$$\frac{d\mathcal{M}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}}{dz} = D^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega)\mathcal{M}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}} + \left[\mathcal{S}_\omega(\mathcal{M})\right]^{\mathbf{t}}_{\mathbf{p},\mathbf{q}} + \left[\mathcal{Z}_\omega(\mathcal{M})\right]^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}\,, \tag{4.2}$$

with the initial conditions $\mathcal{M}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega,z=0;j_1,j_2) = \mathbf{1}_0(|\mathbf{p}|)\mathbf{1}_0(|\mathbf{q}|)\mathbf{1}_{j_1}(l_1)\mathbf{1}_{j_2}(l_2)$. *The linear operator* $\mathcal{Z}_\omega$ *is defined by*

$$\begin{aligned}
\left[\mathcal{Z}_\omega(\mathcal{M})\right]^{\mathbf{t}}_{\mathbf{p},\mathbf{q}} &= \sum_{k=1}^N d^{(4)}_{l_1kl_2}\mathcal{M}^{(k,k)}_{\mathbf{p},\mathbf{q}} \\
&+ \sum_{k_1,k_2=1}^N d^{(2)}_{k_1k_2}\mathcal{M}^{(k_1,k_1)}_{\mathbf{p}\cup\{(k_2,l_1)\},\mathbf{q}\cup\{(k_2,l_2)\}} + d^{(5)}_{k_1k_2}\mathcal{M}^{(k_2,k_1)}_{\mathbf{p}\cup\{(k_1,l_1)\},\mathbf{q}\cup\{(k_2,l_2)\}} \\
&- \sum_{(j,l)\in\mathbf{p}} d^{(6)}_{l_1j}\mathcal{M}^{(j,l_2)}_{\mathbf{p}|\{(j,l)|(l_1,l)\},\mathbf{q}} + d^{(6)}_{l_1l}\mathcal{M}^{(l,l_2)}_{\mathbf{p}|\{n|(j,l_1)\},\mathbf{q}} \\
&+ \sum_{(j,l)\in\mathbf{q}}\sum_{k=1}^N d^{(4)}_{jkl_1}\mathcal{M}^{(k,l_2)}_{\mathbf{p},\mathbf{q}|\{(j,l)|(k,l)\}} + d^{(4)}_{lkl_1}\mathcal{M}^{(k,l_2)}_{\mathbf{p},\mathbf{q}|\{(j,l)|(j,k)\}} \\
&- \sum_{(j,l)\in\mathbf{q}} d^{(6)}_{jl_2}\mathcal{M}^{(l_1,j)}_{\mathbf{p},\mathbf{q}|\{(j,l)|(l_2,l)\}} + d^{(6)}_{ll_2}\mathcal{M}^{(l_1,l)}_{\mathbf{p},\mathbf{q}|\{(j,l)|(j,l_2)\}} \\
&+ \sum_{(j,l)\in\mathbf{p}}\sum_{k=1}^N d^{(4)}_{jkl_2}\mathcal{M}^{(l_1,k)}_{\mathbf{p}|\{(j,l)|(k,l)\},\mathbf{q}} + d^{(4)}_{lkl_2}\mathcal{M}^{(l_1,k)}_{\mathbf{p}|\{(j,l)|(j,k)\},\mathbf{q}} \\
&- \sum_{(j,l)\in\mathbf{p}} d^{(2)}_{jl}\mathcal{M}^{(j,l_2)}_{\mathbf{p}|\{(j,l)|(l,l_1)\},\mathbf{q}} + d^{(5)}_{jl}\mathcal{M}^{(l,l_2)}_{\mathbf{p}|\{(j,l)|(j,l_1)\},\mathbf{q}}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{(j,l)\in\mathbf{q}} d^{(2)}_{jl}\mathcal{M}^{(l_1,j)}_{\mathbf{p},\mathbf{q}|\{(j,l)|(l,l_2)\}} + d^{(5)}_{jl}\mathcal{M}^{(l_1,l)}_{\mathbf{p},\mathbf{q}|\{(j,l)|(j,l_2)\}}\\
&+ \sum_{(j,l)\in\mathbf{q}} \sum_{k_1,k_2=1}^{N} d^{(2)}_{k_1k_2}\mathcal{M}^{(k_1,l_2)}_{\mathbf{p}\cup\{(k_2,l_1)\},\mathbf{q}|\{(j,l)|(j,k_1),(k_2,l)\}}\\
&+ \sum_{(j,l)\in\mathbf{q}} \sum_{k_1,k_2=1}^{N} d^{(5)}_{k_1k_2}\mathcal{M}^{(k_2,l_2)}_{\mathbf{p}\cup\{(k_1,l_1)\},\mathbf{q}|\{(j,l)|(j,k_1),(k_2,l)\}}\\
&+ \sum_{(j,l)\in\mathbf{p}} \sum_{k_1,k_2=1}^{N} d^{(2)}_{k_1k_2}\mathcal{M}^{(l_1,k_1)}_{\mathbf{p}|\{(j,l)|(j,k_1),(k_2,l)\},\mathbf{q}\cup\{(k_2,l_2)\}}\\
&+ \sum_{(j,l)\in\mathbf{p}} \sum_{k_1,k_2=1}^{N} d^{(5)}_{k_1k_2}\mathcal{M}^{(l_1,k_2)}_{\mathbf{p}|\{(j,l)|(j,k_1),(k_2,l)\},\mathbf{q}\cup\{(k_1,l_2)\}}.
\end{aligned}
$$

The coefficient $D^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega)$ *is defined by*

$$
\begin{aligned}
D^{\mathbf{t}}_{\mathbf{p},\mathbf{q}} = D_{\mathbf{p},\mathbf{q}} &+ i\left(\kappa_{l_1}-\kappa_{l_2}\right) - \Gamma_{kl_1} - \overline{\Gamma_{kl_2}} - \sum_{k=1}^{N}\Big(\widetilde{\Gamma}_{kl_1} + \widetilde{\Gamma}_{l_2k} - 2\check{\Gamma}_{l_1l_2}\mathbf{1}_{l_1\neq l_2}\Big)\\
&- 2\sum_{(j,l)\in\mathbf{p}}\left(\check{\Gamma}_{jl_1}\mathbf{1}_{j\neq l_1} + \check{\Gamma}_{ll_1}\mathbf{1}_{l\neq l_1} - \check{\Gamma}_{jl_2}\mathbf{1}_{j\neq l_2} - \check{\Gamma}_{ll_2}\mathbf{1}_{l\neq l_2}\right)\\
&+ 2\sum_{(j,l)\in\mathbf{q}}\left(\check{\Gamma}_{jl_1}\mathbf{1}_{j\neq l_1} + \check{\Gamma}_{ll_1}\mathbf{1}_{l\neq l_1} - \check{\Gamma}_{jl_2}\mathbf{1}_{j\neq l_2} - \check{\Gamma}_{ll_2}\mathbf{1}_{l\neq l_2}\right).
\end{aligned}
$$

The coefficient $d^{(6)}(\omega)$ *is given by*

$$
d^{(6)}_{jl}(\omega) = 2\Re\big[\widetilde{\Gamma}_{jl}(\omega)\big].
$$

The linear operator $\mathcal{S}_\omega$ *and the coefficients* $D_{\mathbf{p},\mathbf{q}}(\omega)$, $\kappa_l(\omega)$, *and* $d^{(j)}(\omega)$, $j=1,\dots,5$, *are defined in Proposition* 3.1.

4.2. Transmission transport equations. We consider here the two-frequency statistics of the transmitted field. We have the following result that is proved in Appendix B.

PROPOSITION 4.2. *We introduce the joint moments of elements of the reflection matrix at two nearby frequencies along with a pair of elements of the transmission matrix:*

$$
\begin{aligned}
(4.3)\quad \mathcal{U}^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}(\omega,h,z;j_1,j_2) = \mathbb{E}\Bigg[&\mathcal{T}^{\varepsilon}_{j_1l_1}(\omega+\varepsilon^2h/2,z)\overline{\mathcal{T}^{\varepsilon}_{j_2l_2}(\omega-\varepsilon^2h/2,z)}\\
&\times\prod_{(j,l)\in\mathbf{p}}\mathcal{R}^{\varepsilon}_{jl}(\omega+\varepsilon^2h/2,z)\prod_{(m,n)\in\mathbf{q}}\overline{\mathcal{R}^{\varepsilon}_{mn}(\omega-\varepsilon^2h/2,z)}\Bigg]
\end{aligned}
$$

for $\mathbf{t}=(l_1,l_2)$. *The family of Fourier transforms*

$$
(4.4)\qquad \mathcal{W}^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z;j_1,j_2) = \frac{1}{2\pi}\int e^{-ih[\tau-\phi_{\mathbf{p},\mathbf{q}}(\omega)z]}\mathcal{U}^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}(\omega,h,z;j_1,j_2)\,dh
$$

converges as $\varepsilon\to0$ *to the solution* $\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}$ *of the system of transport equations*

$$
(4.5)\qquad \frac{\partial\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}}{\partial z} + \phi^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega)\frac{\partial\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}}{\partial\tau} = D^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega)\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}} + \big[\mathcal{S}_\omega(\mathcal{W})\big]^{\mathbf{t}}_{\mathbf{p},\mathbf{q}} + \big[\mathcal{Z}_\omega(\mathcal{W})\big]^{\mathbf{t}}_{\mathbf{p},\mathbf{q}},
$$

with the initial conditions $\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z=0;j_1,j_2) = \mathbf{1}_0(|\mathbf{p}|)\mathbf{1}_0(|\mathbf{q}|)\mathbf{1}_{j_1}(l_1)\mathbf{1}_{j_2}(l_2)\delta(\tau)$. *The coefficient* $\phi^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega)$ *is given by*

$$\phi^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega) = \phi_{\mathbf{p},\mathbf{q}}(\omega) + \frac{\beta'_{l_1}(\omega)+\beta'_{l_2}(\omega)}{2}. \tag{4.6}$$

This generalized set of transport equations describes accurately the transmitted wave field and is the key tool in analyzing various applications with wave propagation in random waveguides. The corresponding transport equations in the layered case are presented in [5].

4.3. Interpretation of the transmission transport equations. We make the following observations regarding the system of transport equations.

(1) By integrating the solution of the system of transport equations in τ, it is straightforward to see that the integral quantity is the solution of the system (4.2). This shows that we have

$$\mathcal{M}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega,z;j_1,j_2) = \int \mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z;j_1,j_2)\,d\tau. \tag{4.7}$$

Therefore, the following remarks stated in terms of the family $\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}$ hold true for the family of moments $\mathcal{M}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}$ as well.

(2) Consider the set of moments $\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}$ such that $|\mathbf{p}|-|\mathbf{q}|=c$ with c a nonzero integer. These moments form a closed subfamily with each member satisfying a zero initial condition. Therefore, these moments vanish, and again only moments having the same number of conjugated and unconjugated terms survive in the small ε limit.

(3) Consider the case (3.13) when $C_{jl}\equiv 0$ for $j\neq l$, as described under (3) in section 3.3. Recall that this corresponds to the situation where modes with different modal wavenumbers are not coupled, which is the case particularly when the inhomogeneities of the waveguide do not have lateral variations. The analysis of the system in (4.2) then shows that the solution has the form

$$\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z;j_1,j_2) = W^{(1)}_{p_1} * \cdots * W^{(l-1)}_{p_{l-1}} * V^{(l)}_{p_l} * W^{(l+1)}_{p_{l+1}} * \cdots * W^{(N)}_{p_N}(\omega,\tau,z)$$

if $\mathbf{t}=(j_1,j_2)=(l,l)$ and $\mathbf{p}=\mathbf{q}=\{(1,1)^{p_1},\dots,(N,N)^{p_N}\}$, and $\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z;j_1,j_2)=0$ otherwise. For each j, $(W^{(j)}_p)_{p\in\mathbb{N}}$ is given by (3.16) and for each l, $(V^{(l)}_p)_{p\in\mathbb{N}}$ is the solution of the closed system of transport equations

$$\frac{\partial V^{(l)}_p}{\partial z} + (2p+1)\beta'_l(\omega)\frac{\partial V^{(l)}_p}{\partial \tau} = 2\Re\big[\Gamma^{(0)}_l(\omega)\big]\left[(p+1)^2(V^{(l)}_{p+1}-V^{(l)}_p) + p^2(V^{(l)}_{p-1}-V^{(l)}_p)\right],$$

with the initial conditions $V^{(l)}_p(\omega,\tau,z=0)=\mathbf{1}_0(p)\delta(\tau)$. We therefore obtain that the backward and forward jth modes are uncoupled from the other modes, but they are coupled together according to the system that governs the propagation of one-dimensional waves in random media [1].

4.4. Forward scattering approximation. To contrast with the fully coupled case discussed above, we address in this section the forward scattering approximation analyzed in detail in [5, 7]. As shown above, this approximation is valid when Γ is zero or very small (in the sense that $\Re(\Gamma)L \ll 1$). The system of transport equations of Proposition 4.2 can be dramatically simplified, since only the terms $\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}$ with $\mathbf{p}=\mathbf{q}=\emptyset$ contribute at the leading order, and these terms satisfy a closed system of transport equations as described in the following proposition.

Proposition 4.3. *If $\Gamma(\omega) = \mathbf{0}$, then the (transformed) autocorrelation function of the transmission coefficients at two nearby frequencies,*

$$\mathcal{V}^{\varepsilon}_{jl}(\omega,\tau,z) = \frac{1}{2\pi}\int e^{-ih[\tau-\beta'_l(\omega)z]}\mathbb{E}\left[\mathcal{T}^{\varepsilon}_{jl}(\omega+\varepsilon^2 h/2,z)\overline{\mathcal{T}^{\varepsilon}_{jl}(\omega-\varepsilon^2 h/2,z)}\right]dh,$$

has a limit as $\varepsilon \to 0$:

$$\lim_{\varepsilon\to 0}\mathcal{V}^{\varepsilon}_{jl}(\omega,\tau,z) = \mathcal{V}_{jl}(\omega,\tau,z).$$

For any fixed $l \in \{1,\dots,N(\omega)\}$, the subfamily $(\mathcal{V}_{jl}(\omega,\tau,z))_{j=1,\dots,N(\omega)}$ is the solution of the system of transport equations

$$\frac{\partial \mathcal{V}_{jl}}{\partial z} + \beta'_j(\omega)\frac{\partial \mathcal{V}_{jl}}{\partial \tau} = \sum_{n\neq j} 2\Re\big[\widetilde{\Gamma}_{jn}(\omega)\big]\left(\mathcal{V}_{nl}-\mathcal{V}_{jl}\right), \tag{4.8}$$

with the initial conditions $\mathcal{V}_{jl}(\omega,\tau,z=0) = \delta(\tau)\mathbf{1}_l(j)$. Here $\widetilde{\Gamma}(\omega)$ is given by (3.4).

Let us introduce the mean transmitted power of the mode j when the input wave is a mode l:

$$\mathcal{P}^{(t)}_{jl}(\omega,z) = \mathcal{M}^{(l,l)}_{\emptyset,\emptyset}(\omega,\tau,z;j,j) = \lim_{\varepsilon\to 0}\mathbb{E}\left[|\mathcal{T}^{\varepsilon}_{jl}(\omega,z)|^2\right].$$

The next proposition shows the equipartition of energy of the transmitted wave.

Proposition 4.4. *If $\Gamma(\omega) = \mathbf{0}$, then the mean transmitted powers converge to the uniform distribution, that is,*

$$\mathcal{P}^{(t)}_{jl}(\omega,z) \stackrel{z\to\infty}{\longrightarrow} \frac{1}{N(\omega)},$$

uniformly in j,l and exponentially in z.

Proof. By integrating (4.8) in τ, we get that, for any fixed l, the subfamily $(\mathcal{P}^{(t)}_{jl}(\omega,z))_{j=1,\dots,N(\omega)}$ is the solution of the linear system

$$\frac{\partial \mathcal{P}^{(t)}_{jl}}{\partial z} = \sum_{n=1}^{N(\omega)} \mathcal{L}^{(t)}_{jn}(\omega)\mathcal{P}^{(t)}_{nl}, \tag{4.9}$$

starting from $\mathcal{P}^{(t)}_{jl}(\omega,z=0) = \mathbf{1}_l(j)$. Here $\mathcal{L}^{(t)}(\omega)$ is the $N(\omega)\times N(\omega)$ matrix

$$\mathcal{L}^{(t)}_{jn}(\omega) = \begin{cases} 2\Re\big[\widetilde{\Gamma}_{jn}(\omega)\big] & \text{if } j\neq n,\\ -2\sum_{m\neq j} 2\Re\big[\widetilde{\Gamma}_{jm}(\omega)\big] & \text{if } j = n.\end{cases}$$

Using the positivity of the coefficients $\Re[\widetilde{\Gamma}_{jn}]$ and the Perron–Frobenius theorem, we find that the matrix $\mathcal{L}^{(t)}$ has zero as an isolated eigenvalue, and all other eigenvalues are negative. It is straightforward to check that the eigenvector corresponding to the zero eigenvalue is the uniform vector, which establishes the proposition. ∎

Appendix A. Derivation of channel reflection-transport equations.

A.1. Propagator equations. We prove here Proposition 3.2. Note first that we can write the first equation in (2.18) in the form

$$\frac{d}{dz}\boldsymbol{\mathcal{R}}^{\varepsilon} = -\boldsymbol{\Phi}^{\varepsilon}\overline{\mathbf{H}^{a,\varepsilon}} + \boldsymbol{\mathcal{R}}^{\varepsilon}\overline{\boldsymbol{\Phi}^{\varepsilon}}\mathbf{H}^{a,\varepsilon}\boldsymbol{\mathcal{R}}^{\varepsilon} + \mathbf{H}^{a,\varepsilon}\boldsymbol{\mathcal{R}}^{\varepsilon} - \boldsymbol{\mathcal{R}}^{\varepsilon}\overline{\mathbf{H}^{a,\varepsilon}}, \tag{A.1}$$

where $\mathbf{\Phi}^\varepsilon(\omega,z)$ is the $N(\omega)\times N(\omega)$ diagonal matrix with diagonal entries:

$$\Phi^\varepsilon_{jj}(\omega,z) = e^{-2i\beta_j(\omega)z/\varepsilon^2}.$$

Our objective is now to compute cross moments of reflection matrix entries using diffusion approximation, and we remark that the phase factors in $\mathbf{\Phi}^\varepsilon$ then act as decoupling terms, decoupling the entries in (A.1). We introduce the quantities $U^\varepsilon_{\mathbf{p},\mathbf{q}}$ that give high-order products of elements $\mathcal{R}^\varepsilon_{jl}$ of the reflection matrix at two nearby frequencies:

$$U^\varepsilon_{\mathbf{p},\mathbf{q}}(\omega,h,z) = \prod_{(j,l)\in\mathbf{p}} \mathcal{R}^\varepsilon_{jl}(\omega+\varepsilon^2 h/2,z) \prod_{(m,n)\in\mathbf{q}} \overline{\mathcal{R}^\varepsilon_{mn}(\omega-\varepsilon^2 h/2,z)}. \tag{A.2}$$

It now follows from (2.18) that the $U^\varepsilon_{\mathbf{p},\mathbf{q}}$'s solve evolution equations of the form

$$\frac{\partial U^\varepsilon_{\mathbf{p},\mathbf{q}}}{\partial z} = \left[\mathcal{H}^\varepsilon_U(U^\varepsilon)\right]_{\mathbf{p},\mathbf{q}}. \tag{A.3}$$

Here $\left[\mathcal{H}^\varepsilon_U(U^\varepsilon)\right]_{\mathbf{p},\mathbf{q}}$ is a finite sum of $U^\varepsilon_{\mathbf{p}^{(1)},\mathbf{q}^{(1)}},\dots,U^\varepsilon_{\mathbf{p}^{(m)},\mathbf{q}^{(m)}}$, where the multi-indices $\mathbf{p}^{(1)},\mathbf{q}^{(1)},\dots,\mathbf{p}^{(N)},\mathbf{q}^{(N)}$ are obtained from $\mathbf{p}$ and $\mathbf{q}$ by one or two replacements. We have explicitly

$$\begin{aligned}
\left[\mathcal{H}^\varepsilon_U(U^\varepsilon)\right]_{\mathbf{p},\mathbf{q}} &= \sum_{(j,l)\in\mathbf{p}} U^\varepsilon_{\mathbf{p}|(j,l),\mathbf{q}} \\
&\quad\times\left\{H^{b,\varepsilon}_{jl} - \sum_{k_1,k_2=1}^N \mathcal{R}^\varepsilon_{jk_1}\overline{H^{b,\varepsilon}_{k_1k_2}}\mathcal{R}^\varepsilon_{k_2l} + \sum_{k=1}^N\left[H^{a,\varepsilon}_{jk}\mathcal{R}^\varepsilon_{kl} - \mathcal{R}^\varepsilon_{jk}\overline{H^{a,\varepsilon}_{kl}}\right]\right\}_{\omega+h\varepsilon^2/2} \\
&\qquad + \sum_{(j,l)\in\mathbf{q}} U^\varepsilon_{\mathbf{p},\mathbf{q}|(j,l)} \\
&\quad\times\overline{\left\{H^{b,\varepsilon}_{jl} - \sum_{k_1,k_2=1}^N \mathcal{R}^\varepsilon_{jk_1}\overline{H^{b,\varepsilon}_{k_1k_2}}\mathcal{R}^\varepsilon_{k_2l} + \sum_{k=1}^N\left[H^{a,\varepsilon}_{jk}\mathcal{R}^\varepsilon_{kl} - \mathcal{R}^\varepsilon_{jk}\overline{H^{a,\varepsilon}_{kl}}\right]\right\}}_{\omega-h\varepsilon^2/2},
\end{aligned}$$

which can also be written as

$$\begin{aligned}
\left[\mathcal{H}^\varepsilon_U(U^\varepsilon)\right]_{\mathbf{p},\mathbf{q}} &= \sum_{(j,l)\in\mathbf{p}}\Bigg\{H^{b,\varepsilon}_{jl}U^\varepsilon_{\mathbf{p}|(j,l),\mathbf{q}} - \sum_{k_1,k_2=1}^N \overline{H^{b,\varepsilon}_{k_1k_2}}U^\varepsilon_{\mathbf{p}|\{(j,l)|(j,k_1),(k_2,l)\},\mathbf{q}} \\
&\qquad + \sum_{k=1}^N\left[H^{a,\varepsilon}_{jk}U^\varepsilon_{\mathbf{p}|\{(j,l)|(k,l)\},\mathbf{q}} - \overline{H^{a,\varepsilon}_{kl}}U^\varepsilon_{\mathbf{p}|\{(j,l)|(j,k)\},\mathbf{q}}\right]\Bigg\}_{\omega+h\varepsilon^2/2} \\
&\qquad + \sum_{(j,l)\in\mathbf{q}}\Bigg\{\overline{H^{b,\varepsilon}_{jl}}U^\varepsilon_{\mathbf{p},\mathbf{q}|(j,l)} - \sum_{k_1,k_2=1}^N H^{b,\varepsilon}_{k_1k_2}U^\varepsilon_{\mathbf{p},\mathbf{q}|\{(j,l)|(j,k_1),(k_2,l)\}} \\
&\qquad + \sum_{k=1}^N\left[\overline{H^{a,\varepsilon}_{jk}}U^\varepsilon_{\mathbf{p},\mathbf{q}|\{(j,l)|(k,l)\}} - H^{a,\varepsilon}_{kl}U^\varepsilon_{\mathbf{p},\mathbf{q}|\{(j,l)|(j,k)\}}\right]\Bigg\}_{\omega-h\varepsilon^2/2}.
\end{aligned} \tag{A.4}$$

Next we observe that

$$
\begin{aligned}
H^{a,\varepsilon}_{jl}\,|_{\omega\pm\varepsilon^2 h/2} &\sim \alpha^{\varepsilon}_{jl}\,(\omega,h,z)\,e^{i(\beta_l(\omega)-\beta_j(\omega))z/\varepsilon^2}e^{\pm i(\beta'_l(\omega)-\beta'_j(\omega))zh/2}\\
&\equiv \alpha^{\pm,\varepsilon}_{jl}\,(\omega,h,z)\,e^{i(\beta_l(\omega)-\beta_j(\omega))z/\varepsilon^2},\\
H^{b,\varepsilon}_{jl}\,|_{\omega\pm\varepsilon^2 h/2} &\sim -\overline{\alpha^{\varepsilon}_{jl}\,(\omega,h,z)}e^{-i(\beta_l(\omega)+\beta_j(\omega))z/\varepsilon^2}e^{\mp i(\beta'_l(\omega)+\beta'_j(\omega))zh/2}\\
&\equiv \widetilde{\alpha}^{\pm,\varepsilon}_{jl}\,(\omega,h,z)\,e^{-i(\beta_l(\omega)+\beta_j(\omega))z/\varepsilon^2}
\end{aligned}
$$

as $\varepsilon\to 0$ for

$$
\begin{aligned}
\alpha^{\varepsilon}_{jl}(\omega,h,z) &= \frac{ik^2(\omega)}{2\varepsilon}\frac{C_{jl}\left(\frac{z}{\varepsilon^2}\right)}{\sqrt{\beta_j\beta_l(\omega)}}\\
&\quad+\frac{ik^4(\omega)}{4}\sum_{l'>N(\omega)}\int_{-\infty}^{\infty}\frac{C_{jl'}\left(\frac{z}{\varepsilon^2}\right)C_{ll'}\left(\frac{z}{\varepsilon^2}+s\right)}{\sqrt{\beta_j\beta_{l'}^2\beta_l(\omega)}}e^{i\beta_l(\omega)s-\beta_{l'}(\omega)|s|}\,ds.
\end{aligned}
$$

Using this notation, we get from (A.4)

$$
\begin{aligned}
\left[\mathcal{H}^{\varepsilon}_U(U^{\varepsilon})\right]_{\mathbf{p},\mathbf{q}} &= \sum_{(j,l)\in\mathbf{p}}\Bigg\{\widetilde{\alpha}^{+,\varepsilon}_{jl}U^{\varepsilon}_{\mathbf{p}|(j,l),\mathbf{q}}e^{-i(\beta_l+\beta_j)z/\varepsilon^2}\\
&\quad+\sum_{k=1}^{N}\left[\alpha^{+,\varepsilon}_{jk}U^{\varepsilon}_{\mathbf{p}|\{(j,l)|(k,l)\},\mathbf{q}}e^{i(\beta_k-\beta_j)z/\varepsilon^2}-\overline{\alpha^{+,\varepsilon}_{kl}}U^{\varepsilon}_{\mathbf{p}|\{(j,l)|(j,k)\},\mathbf{q}}e^{i(\beta_k-\beta_l)z/\varepsilon^2}\right]\\
&\quad-\sum_{k_1,k_2=1}^{N}\overline{\widetilde{\alpha}^{+,\varepsilon}_{k_1k_2}}U^{\varepsilon}_{\mathbf{p}|\{(j,l)|(j,k_1),(k_2,l)\},\mathbf{q}}e^{i(\beta_{k_1}+\beta_{k_2})z/\varepsilon^2}\Bigg\}\\
&\quad+\sum_{(j,l)\in\mathbf{q}}\Bigg\{\overline{\widetilde{\alpha}^{-,\varepsilon}_{jl}}U^{\varepsilon}_{\mathbf{p},\mathbf{q}|(j,l)}e^{i(\beta_j+\beta_l)z/\varepsilon^2}\\
&\quad+\sum_{k=1}^{N}\left[\overline{\alpha^{-,\varepsilon}_{jk}}U^{\varepsilon}_{\mathbf{p},\mathbf{q}|\{(j,l)|(k,l)\}}e^{i(\beta_j-\beta_k)z/\varepsilon^2}-\alpha^{-,\varepsilon}_{kl}U^{\varepsilon}_{\mathbf{p},\mathbf{q}|\{(j,l)|(j,k)\}}e^{i(\beta_l-\beta_k)z/\varepsilon^2}\right]\\
&\quad-\sum_{k_1,k_2=1}^{N}\widetilde{\alpha}^{-,\varepsilon}_{k_1k_2}U^{\varepsilon}_{\mathbf{p},\mathbf{q}|\{(j,l)|(j,k_1),(k_2,l)\}}e^{-i(\beta_{k_1}+\beta_{k_2})z/\varepsilon^2}\Bigg\},
\end{aligned}
\tag{A.5}
$$

where the β_j's are evaluated at ω.

A.2. The homogeneous propagator equations. In order the eliminate the h-dependence in the coefficients of (A.5), we now introduce the transformation

$$
V^{\varepsilon}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z)=\frac{1}{2\pi}\int e^{-ih[\tau-\phi_{\mathbf{p},\mathbf{q}}(\omega)z]}U^{\varepsilon}_{\mathbf{p},\mathbf{q}}(\omega,h,z)\,dh,
\tag{A.6}
$$

where $\phi_{\mathbf{p},\mathbf{q}}(\omega)$ is given by (3.11). We then obtain from (A.5) that V^{ε} solves the infinite-dimensional system of partial differential equations

$$
\frac{\partial V^{\varepsilon}_{\mathbf{p},\mathbf{q}}}{\partial z}+\phi_{\mathbf{p},\mathbf{q}}(\omega)\frac{\partial V^{\varepsilon}_{\mathbf{p},\mathbf{q}}}{\partial\tau}=\left[\mathcal{H}^{\varepsilon}_V(V^{\varepsilon})\right]_{\mathbf{p},\mathbf{q}},
$$

with the initial conditions $V^{\varepsilon}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z=0)=\mathbf{1}_0(|\mathbf{p}|)\mathbf{1}_0(|\mathbf{q}|)\delta(\tau)$. The source term now has the form

$$
\begin{aligned}
\big[\mathcal{H}_V^\varepsilon(V^\varepsilon)\big]_{\mathbf{p},\mathbf{q}} = \sum_{(j,l)\in\mathbf{p}} \Big\{ & -\overline{\alpha_{jl}^\varepsilon} V^\varepsilon_{\mathbf{p}|(j,l),\mathbf{q}} e^{-i(\beta_j+\beta_l)z/\varepsilon^2} \\
& + \sum_{k_1,k_2=1}^N \alpha^\varepsilon_{k_1k_2} V^\varepsilon_{\mathbf{p}|\{(j,l)|(j,k_1),(k_2,l)\},\mathbf{q}} e^{i(\beta_{k_1}+\beta_{k_2})z/\varepsilon^2} \\
& + \sum_{k=1}^N \alpha^\varepsilon_{jk} V^\varepsilon_{\mathbf{p}|\{(j,l)|(k,l)\},\mathbf{q}} e^{i(\beta_k-\beta_j)z/\varepsilon^2} \\
& - \sum_{k=1}^N \overline{\alpha^\varepsilon_{kl}} V^\varepsilon_{\mathbf{p}|\{(j,l)|(j,k)\},\mathbf{q}} e^{i(\beta_k-\beta_l)z/\varepsilon^2} \Big\} \\
& + \sum_{(j,l)\in\mathbf{q}} \Big\{ -\alpha^\varepsilon_{jl} V^\varepsilon_{\mathbf{p},\mathbf{q}|(j,l)} e^{i(\beta_j+\beta_l)z/\varepsilon^2} \\
& + \sum_{k_1,k_2=1}^N \overline{\alpha^\varepsilon_{k_1k_2}} V^\varepsilon_{\mathbf{p},\mathbf{q}|\{(j,l)|(j,k_1),(k_2,l)\}} e^{-i(\beta_{k_1}+\beta_{k_2})z/\varepsilon^2} \\
& + \sum_{k=1}^N \overline{\alpha^\varepsilon_{jk}} V^\varepsilon_{\mathbf{p},\mathbf{q}|\{(j,l)|(k,l)\}} e^{i(\beta_j-\beta_k)z/\varepsilon^2} \\
& - \sum_{k=1}^N \alpha^\varepsilon_{kl} V^\varepsilon_{\mathbf{p},\mathbf{q}|\{(j,l)|(j,k)\}} e^{i(\beta_l-\beta_k)z/\varepsilon^2} \Big\},
\end{aligned}
\tag{A.7}
$$

where the β_j's are evaluated at ω.

A.3. Transport equations. We next apply the diffusion approximation to get transport equations for the moments; see [5] for background material on and related to applications of the diffusion approximation theory. Observe that the function $\mathcal{H}_V^\varepsilon$ is linear and the random coefficients are rapidly fluctuating. Those coefficients whose amplitudes are of order ε^{-1} are centered and fluctuate on the scale ε^2; moreover, they are assumed to be rapidly mixing, giving a white-noise scaling situation. We can thus apply diffusion approximation results to obtain transport equations for the moments $\mathbb{E}[V^\varepsilon_{\mathbf{p},\mathbf{q}}]$ in the limit $\varepsilon \to 0$:

$$
\mathcal{W}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z) = \lim_{\varepsilon\to 0} \mathbb{E}[V^\varepsilon_{\mathbf{p},\mathbf{q}}(\omega,\tau,z)].
$$

We then obtain from (A.7) that $\mathcal{W}_{\mathbf{p},\mathbf{q}}$ solves the infinite-dimensional system of partial differential equations

$$
\frac{\partial \mathcal{W}_{\mathbf{p},\mathbf{q}}}{\partial z} + \phi_{\mathbf{p},\mathbf{q}}(\omega)\frac{\partial \mathcal{W}_{\mathbf{p},\mathbf{q}}}{\partial \tau} = i\Big[\sum_{(j,l)\in\mathbf{p}}\big(\kappa_j+\kappa_l\big) - \sum_{(j,l)\in\mathbf{q}}\big(\kappa_j+\kappa_l\big)\Big]\mathcal{W}_{\mathbf{p},\mathbf{q}} + \big[\mathcal{H}(\mathcal{W})\big]_{\mathbf{p},\mathbf{q}},
$$

with the initial conditions $\mathcal{W}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z=0) = \mathbf{1}_0(|\mathbf{p}|)\mathbf{1}_0(|\mathbf{q}|)\delta(\tau)$, and where we defined $\kappa_l(\omega)$ (which is real) by (3.6). The source term now takes the form

$$
\big[\mathcal{H}(\mathcal{W})\big]_{\mathbf{p},\mathbf{q}} = \sum_{k=1}^6 \mathcal{I}_k,
\tag{A.8}
$$

and we next identify the coupling terms $\mathcal{I}_k$. We remark that in applying the diffusion approximation there is no coupling between terms that contain phase modulation of the type $\exp[i(\beta_j-\beta_l)z/\varepsilon^2]$ with terms that contain phase modulation of the type $\exp[i(\beta_m+\beta_n)z/\varepsilon^2]$ since the rapid phases then cannot cancel. There are eight terms in the expression for $\mathcal{H}_V^\varepsilon$ in (A.7); we label the first four associated with the multi-index $\mathbf{p}$ by $1_p, \dots, 4_p$ and the last by four by $1_q, \dots, 4_q$. First we consider the cross interaction of the terms 1_p and 2_p and also the corresponding combination 1_q and 2_q that is associated with complex conjugate coefficients. We label their contribution by the term $\mathcal{I}_1$, which is given by

$$\begin{aligned}
\mathcal{I}_1 = &- \sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{p}} 2\Re\big(\Gamma_{jl}\big) \Big(\mathcal{W}_{\mathbf{p}|\{(j,l),(\tilde{j},\tilde{l})|(\tilde{j},j),(l,\tilde{l})\},\mathbf{q}} + \mathcal{W}_{\mathbf{p}|\{(j,l),(\tilde{j},\tilde{l})|(\tilde{j},l),(j,\tilde{l})\},\mathbf{q}}\mathbf{1}_{j\neq l}\Big) \\
&- \sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{p}} 2\Re\big(\Gamma_{\tilde{j}\tilde{l}}\big) \Big(\mathcal{W}_{\mathbf{p}|\{(j,l),(\tilde{j},\tilde{l})|(j,\tilde{j}),(\tilde{l},l)\},\mathbf{q}} + \mathcal{W}_{\mathbf{p}|\{(j,l),(\tilde{j},\tilde{l})|(l,\tilde{j}),(\tilde{l},j)\},\mathbf{q}}\mathbf{1}_{\tilde{j}\neq \tilde{l}}\Big) \\
&- \sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{q}} 2\Re\big(\Gamma_{jl}\big) \Big(\mathcal{W}_{\mathbf{p},\mathbf{q}|\{(j,l),(\tilde{j},\tilde{l})|(\tilde{j},j),(l,\tilde{l})\}} + \mathcal{W}_{\mathbf{p},\mathbf{q}|\{(j,l),(\tilde{j},\tilde{l})|(\tilde{j},l),(j,\tilde{l})\}}\mathbf{1}_{j\neq l}\Big) \\
&- \sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{q}} 2\Re\big(\Gamma_{\tilde{j}\tilde{l}}\big) \Big(\mathcal{W}_{\mathbf{p},\mathbf{q}|\{(j,l),(\tilde{j},\tilde{l})|(j,\tilde{j}),(\tilde{l},l)\}} + \mathcal{W}_{\mathbf{p},\mathbf{q}|\{(j,l),(\tilde{j},\tilde{l})|(l,\tilde{j}),(\tilde{l},j)\}}\mathbf{1}_{\tilde{j}\neq \tilde{l}}\Big) \\
&- \sum_{(j,l)\in\mathbf{p}} \sum_{k=1}^N \left(\Gamma_{jk} + \Gamma_{lk}\right) \mathcal{W}_{\mathbf{p},\mathbf{q}} - \sum_{(j,l)\in\mathbf{q}} \sum_{k=1}^N \left(\overline{\Gamma_{jk}} + \overline{\Gamma_{lk}}\right) \mathcal{W}_{\mathbf{p},\mathbf{q}},
\end{aligned}$$

where Γ is defined by (3.5).

Next we consider the cross interaction of the terms 1_p and 2_p with the terms 1_q and 2_q. We label their contribution by the term $\mathcal{I}_2$, which is given by

$$\begin{aligned}
\mathcal{I}_2 = &\sum_{(j,l)\in\mathbf{p}} \sum_{(\tilde{j},\tilde{l})\in\mathbf{q}} 2\Re\big(\Gamma_{jl}\big)\mathcal{W}_{\mathbf{p}|(j,l),\mathbf{q}|(\tilde{j},\tilde{l})}\mathbf{1}_{(j,l)\cong(\tilde{j},\tilde{l})} \\
&+ \sum_{(j,l)\in\mathbf{p}} \sum_{(\tilde{j},\tilde{l})\in\mathbf{q}} \sum_{k_1,k_2=1}^N 2\Re\big(\Gamma_{k_1k_2}\big)\mathcal{W}_{\mathbf{p}|\{(j,l)|(j,k_1),(k_2,l)\},\mathbf{q}|\{(\tilde{j},\tilde{l})|(\tilde{j},k_1),(k_2,\tilde{l})\}} \\
&+ \sum_{(j,l)\in\mathbf{p}} \sum_{(\tilde{j},\tilde{l})\in\mathbf{q}} \sum_{k_1,k_2=1}^N 2\Re\big(\Gamma_{k_1k_2}\big)\mathcal{W}_{\mathbf{p}|\{(j,l)|(j,k_1),(k_2,l)\},\mathbf{q}|\{(\tilde{j},\tilde{l})|(\tilde{j},k_2),(k_1,\tilde{l})\}}\mathbf{1}_{k_1\neq k_2},
\end{aligned}$$

where $(j,l) \cong (\tilde{j},\tilde{l})$ if $(j,l) = (\tilde{j},\tilde{l})$ or $(j,l) = (\tilde{l},\tilde{j})$.

We have completed the analysis of the terms associated with phase modulation of the form $\exp[i(\beta_j + \beta_l)z/\varepsilon^2]$ and consider now terms associated with phases of the form $\exp[i(\beta_j - \beta_l)z/\varepsilon^2]$. Consider first the interaction of the terms 3_p, 4_p, 3_q, and 4_q with themselves. We label this contribution by $\mathcal{I}_3$, which is given by

$$\begin{aligned}
\mathcal{I}_3 = &-\Big\{2\sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{p}} \Big[\check{\Gamma}_{j\tilde{j}} + \check{\Gamma}_{l\tilde{l}}\Big] + \sum_{(j,l)\in\mathbf{p}} \sum_{k=1}^N \Big[\overline{\widetilde{\Gamma}_{jk}} + \overline{\widetilde{\Gamma}_{lk}}\Big]\Big\}\mathcal{W}_{\mathbf{p},\mathbf{q}} \\
&- 2\sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{p}} \Big[\mathbf{1}_{j\neq\tilde{j}}\Re\big(\widetilde{\Gamma}_{j\tilde{j}}\big) + \mathbf{1}_{l\neq\tilde{l}}\Re\big(\widetilde{\Gamma}_{l\tilde{l}}\big)\Big]\mathcal{W}_{\mathbf{p}|\{(j,l),(\tilde{j},\tilde{l})|(\tilde{j},l),(j,\tilde{l})\},\mathbf{q}} \\
&-\Big\{2\sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{q}} \Big[\check{\Gamma}_{j\tilde{j}} + \check{\Gamma}_{l\tilde{l}}\Big] + \sum_{(j,l)\in\mathbf{q}} \sum_{k=1}^N \Big[\widetilde{\Gamma}_{jk} + \widetilde{\Gamma}_{lk}\Big]\Big\}\mathcal{W}_{\mathbf{p},\mathbf{q}} \\
&- 2\sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{q}} \Big[\mathbf{1}_{j\neq\tilde{j}}\Re\big(\widetilde{\Gamma}_{j\tilde{j}}\big) + \mathbf{1}_{l\neq\tilde{l}}\Re\big(\widetilde{\Gamma}_{l\tilde{l}}\big)\Big]\mathcal{W}_{\mathbf{p},\mathbf{q}|\{(j,l),(\tilde{j},\tilde{l})|(\tilde{j},l),(j,\tilde{l})\}},
\end{aligned}$$

where $\check{\Gamma}$ and $\widetilde{\Gamma}$ are defined by (3.3)–(3.4).

Next, we deal with the cross interaction between the terms 3_p and 4_p and correspondingly between 3_q and 4_q. We label this contribution by $\mathcal{I}_4$ and obtain

$$\begin{aligned}
\mathcal{I}_4 = &-\Big\{\sum_{(j,l)\in\mathbf{p}}\sum_{(\tilde{j},\tilde{l})\in\mathbf{p}} 2\check{\Gamma}_{j\tilde{l}}\Big\}\mathcal{W}_{\mathbf{p},\mathbf{q}} - \sum_{(j,l)\in\mathbf{p}} 2\Re\big(\widetilde{\Gamma}_{jl}\big)\mathcal{W}_{\mathbf{p}|\{(j,l)|(l,j)\},\mathbf{q}}\mathbf{1}_{j\neq l}\\
&-2\sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{p}} \Re\big(\widetilde{\Gamma}_{j\tilde{l}}\big)\mathbf{1}_{j\neq\tilde{l}}\mathcal{W}_{\mathbf{p}|\{(j,l),(\tilde{j},\tilde{l})|(\tilde{l},l),(\tilde{j},j)\},\mathbf{q}}\\
&-2\sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{p}} \Re\big(\widetilde{\Gamma}_{\tilde{j}l}\big)\mathbf{1}_{\tilde{j}\neq l}\mathcal{W}_{\mathbf{p}|\{(j,l),(\tilde{j},\tilde{l})|(l,\tilde{l}),(j,\tilde{j})\},\mathbf{q}}\\
&-\Big\{\sum_{(j,l)\in\mathbf{q}}\sum_{(\tilde{j},\tilde{l})\in\mathbf{q}} 2\check{\Gamma}_{j\tilde{l}}\Big\}\mathcal{W}_{\mathbf{p},\mathbf{q}} - \sum_{(j,l)\in\mathbf{q}} 2\Re\big(\widetilde{\Gamma}_{jl}\big)\mathcal{W}_{\mathbf{p},\mathbf{q}|\{(j,l)|(l,j)\}}\mathbf{1}_{j\neq l}\\
&-2\sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{q}} \Re\big(\widetilde{\Gamma}_{j\tilde{l}}\big)\mathbf{1}_{j\neq\tilde{l}}\mathcal{W}_{\mathbf{p},\mathbf{q}|\{(j,l),(\tilde{j},\tilde{l})|(\tilde{l},l),(\tilde{j},j)\}}\\
&-2\sum_{\{(j,l),(\tilde{j},\tilde{l})\}\in\mathbf{q}} \Re\big(\widetilde{\Gamma}_{\tilde{j}l}\big)\mathbf{1}_{\tilde{j}\neq l}\mathcal{W}_{\mathbf{p},\mathbf{q}|\{(j,l),(\tilde{j},\tilde{l})|(l,\tilde{l}),(j,\tilde{j})\}}.
\end{aligned}$$

Now we consider the cross interaction between the terms 3_p and 3_q and correspondingly between 4_p and 4_q. We label this contribution by $\mathcal{I}_5$ and obtain

$$\begin{aligned}
\mathcal{I}_5 = &\sum_{(j,l)\in\mathbf{p}}\sum_{(\tilde{j},\tilde{l})\in\mathbf{q}}\left[2\check{\Gamma}_{\tilde{j}j}\mathcal{W}_{\mathbf{p},\mathbf{q}} + \sum_{k=1\neq j}^{N} 2\Re\left[\widetilde{\Gamma}_{jk}\right]\mathcal{W}_{\mathbf{p}|\{(j,l)|(k,l)\},\mathbf{q}|\{(\tilde{j},\tilde{l})|(k,\tilde{l})\}}\mathbf{1}_{j=\tilde{j}}\right]\\
&+\sum_{(j,l)\in\mathbf{p}}\sum_{(\tilde{j},\tilde{l})\in\mathbf{q}}\left[2\check{\Gamma}_{\tilde{l}l}\mathcal{W}_{\mathbf{p},\mathbf{q}} + \sum_{k=1\neq l}^{N} 2\Re\big(\widetilde{\Gamma}_{lk}\big)\mathcal{W}_{\mathbf{p}|\{(j,l)|(j,k)\},\mathbf{q}|\{(\tilde{j},\tilde{l})|(\tilde{j},k)\}}\mathbf{1}_{l=\tilde{l}}\right].
\end{aligned}$$

Finally, we analyze the cross interaction between the terms 3_p and 4_q and correspondingly between 4_p and 3_q. We label this contribution by $\mathcal{I}_6$ and obtain

$$\begin{aligned}
\mathcal{I}_6 = &\sum_{(j,l)\in\mathbf{p}}\sum_{(\tilde{j},\tilde{l})\in\mathbf{q}}\left[2\check{\Gamma}_{j\tilde{l}}\mathcal{W}_{\mathbf{p},\mathbf{q}} + \sum_{k=1\neq j}^{N} 2\Re\big(\widetilde{\Gamma}_{jk}\big)\mathcal{W}_{\mathbf{p}|\{(j,l)|(k,l)\},\mathbf{q}|\{(\tilde{j},\tilde{l})|(\tilde{j},k)\}}\mathbf{1}_{j=\tilde{l}}\right]\\
&+\sum_{(j,l)\in\mathbf{p}}\sum_{(\tilde{j},\tilde{l})\in\mathbf{q}}\left[2\check{\Gamma}_{l\tilde{j}}\mathcal{W}_{\mathbf{p},\mathbf{q}} + \sum_{k=1\neq l}^{N} 2\Re\big(\widetilde{\Gamma}_{lk}\big)\mathcal{W}_{\mathbf{p}|\{(j,l)|(j,k)\},\mathbf{q}|\{(\tilde{j},\tilde{l})|(k,\tilde{l})\}}\mathbf{1}_{l=\tilde{j}}\right].
\end{aligned}$$

We can now assemble the terms in the source term $\mathcal{H}$ for the transport equation, and this completes the proof of Proposition 3.2.

Appendix B. Derivation of channel transmission-transport equations. We consider next the wave field that has been transmitted through the waveguide and develop a family of transport equations that generalize those we derived above for the characterization of the reflected field. The transmitted field can be characterized by the transmission operator in (2.17). Recall that the transmission and reflection matrices solve (2.18). In order to obtain a closed system of transport equations, we introduce the quantities

$$U^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}(\omega,h,z;j_1,j_2) = \mathcal{T}^{\varepsilon}_{j_1l_1}(\omega+\varepsilon^2h/2,z)\overline{\mathcal{T}^{\varepsilon}_{j_2l_2}(\omega-\varepsilon^2h/2,z)}U^{\varepsilon}_{\mathbf{p},\mathbf{q}}(\omega,h,z)$$

for $\mathbf{t} = (l_1, l_2)$. Then we find, using (A.3),

$$\frac{\partial U^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}}{\partial z} = \big[\mathcal{H}^\varepsilon_U(U^{\mathbf{t},\varepsilon})\big]_{\mathbf{p},\mathbf{q}} - U^\varepsilon_{\mathbf{p},\mathbf{q}} \left\{\overline{\mathcal{T}^\varepsilon_{j_2 l_2}}\right\}_{\omega - h\varepsilon^2/2} \left\{\sum_{k_1=1}^N \mathcal{T}^\varepsilon_{j_1 k_1}\left(\overline{H^{a,\varepsilon}_{k_1 l_1}} + \sum_{k_2=1}^N \overline{H^{b,\varepsilon}_{k_1 k_2}}\mathcal{R}^\varepsilon_{k_2 l_1}\right)\right\}_{\omega + h\varepsilon^2/2} - U^\varepsilon_{\mathbf{p},\mathbf{q}} \left\{\mathcal{T}^\varepsilon_{j_1 l_1}\right\}_{\omega + h\varepsilon^2/2} \left\{\sum_{k_1=1}^N \overline{\mathcal{T}^\varepsilon_{j_2 k_1}}\left(H^{a,\varepsilon}_{k_1 l_2} + \sum_{k_2=1}^N H^{b,\varepsilon}_{k_1 k_2}\overline{\mathcal{R}^\varepsilon_{k_2 l_2}}\right)\right\}_{\omega - h\varepsilon^2/2},$$

with $\mathcal{H}^\varepsilon_U$ defined in (A.5). We remark that the family of coefficients $U^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}(\omega, h, z; j_1, j_2)$ for fixed j_1 and j_2 form a closed subfamily, which allows us to rewrite the previous system as

$$\frac{\partial U^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}}{\partial z} = \big[\mathcal{H}^\varepsilon_U(U^{\mathbf{t},\varepsilon})\big]_{\mathbf{p},\mathbf{q}} + \big[\mathcal{H}^{\varepsilon,1}_U(U^\varepsilon)\big]^{\mathbf{t}}_{\mathbf{p},\mathbf{q}} + \big[\mathcal{H}^{\varepsilon,2}_U(U^\varepsilon)\big]^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}, \tag{B.1}$$

$$\big[\mathcal{H}^{\varepsilon,1}_U(U^\varepsilon)\big]^{\mathbf{t}}_{\mathbf{p},\mathbf{q}} = -\sum_{k=1}^N \left(\left\{\overline{H^{a,\varepsilon}_{k l_1}}\right\}_{\omega + h\varepsilon^2/2} U^{(k,l_2),\varepsilon}_{\mathbf{p},\mathbf{q}} + \left\{H^{a,\varepsilon}_{k l_2}\right\}_{\omega - h\varepsilon^2/2} U^{(l_1,k),\varepsilon}_{\mathbf{p},\mathbf{q}}\right),$$

$$\big[\mathcal{H}^{\varepsilon,2}_U(U^\varepsilon)\big]^{\mathbf{t}}_{\mathbf{p},\mathbf{q}} = -\sum_{k_1,k_2=1}^N \left(\left\{\mathcal{R}^\varepsilon_{k_2 l_1}\overline{H^{b,\varepsilon}_{k_1 k_2}}\right\}_{\omega + h\varepsilon^2/2} U^{(k_1,l_2),\varepsilon}_{\mathbf{p},\mathbf{q}} + \left\{\overline{\mathcal{R}^\varepsilon_{k_2 l_2}} H^{b,\varepsilon}_{k_1 k_2}\right\}_{\omega - h\varepsilon^2/2} U^{(l_1,k_1),\varepsilon}_{\mathbf{p},\mathbf{q}}\right).$$

B.1. Homogeneous propagator equations in the transmission case. In order the eliminate the h-dependence in the coefficients of (B.1), we introduce the transformation

$$V^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}(\omega, \tau, z; j_1, j_2) = \frac{1}{2\pi}\int e^{-ih[\tau - \phi^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega) z]} U^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}(\omega, h, z; j_1, j_2)\, dh, \tag{B.2}$$

with $\phi^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega)$ defined in (4.6). We then obtain from (B.1) that $V^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}$ solves the infinite-dimensional system of partial differential equations

$$\frac{\partial V^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}}{\partial z} + \phi^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega)\frac{\partial V^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}}{\partial \tau} = \big[\widetilde{\mathcal{H}}^\varepsilon_V(V^\varepsilon)\big]^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}, \tag{B.3}$$

with the initial conditions $V^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}(\omega, \tau, z = 0; j_1, j_2) = \mathbf{1}_0(|\mathbf{p}|)\mathbf{1}_0(|\mathbf{q}|)\mathbf{1}_{j_1}(l_1)\mathbf{1}_{j_2}(l_2)\delta(\tau)$. We decompose the source term as

$$\widetilde{\mathcal{H}}^\varepsilon_V = \mathcal{H}^\varepsilon_V + \mathcal{H}^{\varepsilon,1}_V + \mathcal{H}^{\varepsilon,2}_V, \tag{B.4}$$

with $\mathcal{H}^\varepsilon_V$ defined in (A.7) and the specific transmission source terms given by

(B.5)

$$\big[\mathcal{H}^{\varepsilon,1}_V(V^\varepsilon)\big]^{\mathbf{t}}_{\mathbf{p},\mathbf{q}} = -\sum_{k=1}^N \left[\overline{\alpha^\varepsilon_{k l_1}} V^{(k,l_2),\varepsilon}_{\mathbf{p},\mathbf{q}} e^{i(\beta_k - \beta_{l_1})z/\varepsilon^2} + \alpha^\varepsilon_{k l_2} V^{(l_1,k),\varepsilon}_{\mathbf{p},\mathbf{q}} e^{i(\beta_{l_2} - \beta_k)z/\varepsilon^2}\right],$$

$$\big[\mathcal{H}^{\varepsilon,2}_V(V^\varepsilon)\big]^{\mathbf{t}}_{\mathbf{p},\mathbf{q}} = \sum_{k_1,k_2=1}^N \left[\alpha^\varepsilon_{k_1 k_2} V^{(k_1,l_2),\varepsilon}_{\mathbf{p}\cup\{(k_2,l_1)\},\mathbf{q}} e^{i(\beta_{k_1} + \beta_{k_2})z/\varepsilon^2} + \overline{\alpha^\varepsilon_{k_1 k_2}} V^{(l_1,k_1),\varepsilon}_{\mathbf{p},\mathbf{q}\cup\{(k_2,l_2)\}} e^{-i(\beta_{k_1} + \beta_{k_2})z/\varepsilon^2}\right], \tag{B.6}$$

where the β_j's are evaluated at ω.

B.2. Transport equations. We now apply the diffusion approximation to get transport equations for the above modified moments that are relevant in the transmission case. That is, we deduce transport equations for the moments $\mathbb{E}[V^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}]$ in the limit $\varepsilon \to 0$:

$$\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z;j_1,j_2) = \lim_{\varepsilon\to 0} \mathbb{E}[V^{\mathbf{t},\varepsilon}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z;j_1,j_2)].$$

We then obtain from (B.3) that $\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}$ solves the infinite-dimensional system of partial differential equations

$$\begin{aligned}\frac{\partial \mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}}{\partial z} + \phi^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega)\frac{\partial \mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}}{\partial \tau} &= i\Big[\kappa_{l_1} - \kappa_{l_2} + \sum_{(j,l)\in\mathbf{p}}\big(\kappa_j+\kappa_l\big) - \sum_{(j,l)\in\mathbf{q}}\big(\kappa_j+\kappa_l\big)\Big]\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}\\ &\quad + \big[\mathcal{H}(\mathcal{W}^{\mathbf{t}})\big]_{\mathbf{p},\mathbf{q}} + \big[\mathcal{H}^1(\mathcal{W})\big]^{\mathbf{t}}_{\mathbf{p},\mathbf{q}},\end{aligned}$$

with the initial conditions $\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}(\omega,\tau,z=0;j_1,j_2) = \mathbf{1}_0(|\mathbf{p}|)\mathbf{1}_0(|\mathbf{q}|)\mathbf{1}_{j_1}(l_1)\mathbf{1}_{j_2}(l_2)\delta(\tau)$. The source term $\mathcal{H}$ is defined in (A.8), and the specific transmission source term has the form

$$\big[\mathcal{H}^1(\mathcal{W})\big]^{\mathbf{t}}_{\mathbf{p},\mathbf{q}} = \sum_{k=1}^{4}\widetilde{\mathcal{I}}_k, \tag{B.7}$$

and we next identify the coupling terms $\widetilde{\mathcal{I}}_k$.

First, we consider the terms that correspond to the interaction of the terms $\mathcal{H}^{\varepsilon,1}_V$ in (B.5) with themselves. This contribution is

$$\widetilde{\mathcal{I}}_1 = 2\sum_{k=1}^{N}\Re\big(\widetilde{\Gamma}_{kl_1}\big)\mathcal{W}^{(k,k)}_{\mathbf{p},\mathbf{q}}\mathbf{1}_{l_1=l_2} - \sum_{k=1}^{N}\Big[\widetilde{\Gamma}_{kl_1} + \widetilde{\Gamma}_{l_2k} - 2\check{\Gamma}_{l_1l_2}\Big]\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}\mathbf{1}_{l_1\neq l_2}.$$

Then, we consider the cross interaction of the terms in $\mathcal{H}^{\varepsilon,2}_V$ in (B.5). This gives the contribution

$$\widetilde{\mathcal{I}}_2 = 2\sum_{k_1,k_2=1}^{N}\Re\big(\Gamma_{k_1k_2}\big)\Big[\mathcal{W}^{(k_1,k_1)}_{\mathbf{p}\cup\{(k_2,l_1)\},\mathbf{q}\cup\{(k_2,l_2)\}} + \mathcal{W}^{(k_2,k_1)}_{\mathbf{p}\cup\{(k_1,l_1)\},\mathbf{q}\cup\{(k_2,l_2)\}}\mathbf{1}_{k_2\neq k_1}\Big].$$

The terms in $\mathcal{H}^{\varepsilon,1}_V$ interact with those in $\mathcal{H}^{\varepsilon}_V$ having phase modulations of the form $\exp[i(\beta_j-\beta_l)z/\varepsilon^2]$, giving the following contribution to the diffusion approximation:

$$\begin{aligned}\widetilde{\mathcal{I}}_3 = &-2\sum_{(j,l)\in\mathbf{p}}\big[\check{\Gamma}_{jl_1}\mathbf{1}_{j\neq l_1} + \check{\Gamma}_{ll_1}\mathbf{1}_{l\neq l_1} - \check{\Gamma}_{jl_2}\mathbf{1}_{j\neq l_2} - \check{\Gamma}_{ll_2}\mathbf{1}_{l\neq l_2}\big]\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}\\ &+2\sum_{(j,l)\in\mathbf{q}}\big[\check{\Gamma}_{jl_1}\mathbf{1}_{j\neq l_1} + \check{\Gamma}_{ll_1}\mathbf{1}_{l\neq l_1} - \check{\Gamma}_{jl_2}\mathbf{1}_{j\neq l_2} - \check{\Gamma}_{ll_2}\mathbf{1}_{l\neq l_2}\big]\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}}\\ &-2\sum_{(j,l)\in\mathbf{p}}\Big[\Re\big(\widetilde{\Gamma}_{l_1j}\big)\mathcal{W}^{(j,l_2)}_{\mathbf{p}|\{(j,l)|(l_1,l)\},\mathbf{q}} + \Re\big(\widetilde{\Gamma}_{l_1l}\big)\mathcal{W}^{(l,l_2)}_{\mathbf{p}|\{(j,l)|(j,l_1)\},\mathbf{q}}\Big]\\ &+2\sum_{(j,l)\in\mathbf{q}}\sum_{k=1}^{N}\Big[\Re\big(\widetilde{\Gamma}_{jk}\big)\mathcal{W}^{(k,l_2)}_{\mathbf{p},\mathbf{q}|\{(j,l)|(k,l)\}}\mathbf{1}_{j=l_1} + \Re\big(\widetilde{\Gamma}_{lk}\big)\mathcal{W}^{(k,l_2)}_{\mathbf{p},\mathbf{q}|\{(j,l)|(j,k)\}}\mathbf{1}_{l=l_1}\Big]\\ &-2\sum_{(j,l)\in\mathbf{q}}\Big[\Re\big(\widetilde{\Gamma}_{jl_2}\big)\mathcal{W}^{(l_1,j)}_{\mathbf{p},\mathbf{q}|\{(j,l)|(l_2,l)\}} + \Re\big(\widetilde{\Gamma}_{ll_2}\big)\mathcal{W}^{(l_1,l)}_{\mathbf{p},\mathbf{q}|\{(j,l)|(j,l_2)\}}\Big]\\ &+2\sum_{(j,l)\in\mathbf{p}}\sum_{k=1}^{N}\Big[\Re\big(\widetilde{\Gamma}_{kj}\big)\mathcal{W}^{(l_1,k)}_{\mathbf{p}|\{(j,l)|(k,l)\},\mathbf{q}}\mathbf{1}_{j=l_2} + \Re\big(\widetilde{\Gamma}_{kl}\big)\mathcal{W}^{(l_1,k)}_{\mathbf{p}|\{(j,l)|(j,k)\},\mathbf{q}}\mathbf{1}_{l=l_2}\Big].\end{aligned}$$

Finally, we consider the cross interaction of the terms in $\mathcal{H}_V^{\varepsilon,2}$ with those in $\mathcal{H}_V^{\varepsilon}$. This gives the contribution

$$
\begin{aligned}
\widetilde{\mathcal{I}}_4 = &-\sum_{k=1}^N \Gamma_{kl_1}\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}} - 2\sum_{(j,l)\in\mathbf{p}} \Re\big(\Gamma_{jl}\big)\left[\mathcal{W}^{(j,l_2)}_{\mathbf{p}|\{(j,l)|(l,l_1)\},\mathbf{q}} + \mathcal{W}^{(l,l_2)}_{\mathbf{p}|\{(j,l)|(j,l_1)\},\mathbf{q}}\mathbf{1}_{j\neq l}\right] \\
&-\sum_{k=1}^N \overline{\Gamma_{kl_2}}\mathcal{W}^{\mathbf{t}}_{\mathbf{p},\mathbf{q}} - 2\sum_{(j,l)\in\mathbf{q}} \Re\big(\Gamma_{jl}\big)\left[\mathcal{W}^{(l_1,j)}_{\mathbf{p},\mathbf{q}|\{(j,l)|(l,l_2)\}} + \mathcal{W}^{(l_1,l)}_{\mathbf{p},\mathbf{q}|\{(j,l)|(j,l_2)\}}\mathbf{1}_{j\neq l}\right] \\
&+2\sum_{(j,l)\in\mathbf{q}}\sum_{k_1,k_2=1}^N \Re\big(\Gamma_{k_1k_2}\big)\mathcal{W}^{(k_1,l_2)}_{\mathbf{p}\cup\{(k_2,l_1)\},\mathbf{q}|\{(j,l)|(j,k_1),(k_2,l)\}} \\
&+2\sum_{(j,l)\in\mathbf{q}}\sum_{k_1,k_2=1}^N \Re\big(\Gamma_{k_1k_2}\big)\mathcal{W}^{(k_2,l_2)}_{\mathbf{p}\cup\{(k_1,l_1)\},\mathbf{q}|\{(j,l)|(j,k_1),(k_2,l)\}}\mathbf{1}_{k_1\neq k_2} \\
&+2\sum_{(j,l)\in\mathbf{p}}\sum_{k_1,k_2=1}^N \Re\big(\Gamma_{k_1k_2}\big)\mathcal{W}^{(l_1,k_1)}_{\mathbf{p}|\{(j,l)|(j,k_1),(k_2,l)\},\mathbf{q}\cup\{(k_2,l_2)\}} \\
&+2\sum_{(j,l)\in\mathbf{p}}\sum_{k_1,k_2=1}^N \Re\big(\Gamma_{k_1k_2}\big)\mathcal{W}^{(l_1,k_2)}_{\mathbf{p}|\{(j,l)|(j,k_1),(k_2,l)\},\mathbf{q}\cup\{(k_1,l_2)\}}\mathbf{1}_{k_1\neq k_2}.
\end{aligned}
$$

We can now assemble the terms in the source term $\mathcal{H}^1$ for the transport equation, and this completes the proof of Proposition 4.2.

REFERENCES

[1] M. Asch, W. Kohler, G. Papanicolaou, M. Postel, and B. White, *Frequency content of randomly scattered signals*, SIAM Rev., 33 (1991), pp. 519–625.

[2] Y. N. Barabanenkov, *Wave corrections for the transfer equation for backward scattering*, Izv. Vyssh. Uchebn. Zaved. Radiofiz., 16 (1973), pp. 88–96.

[3] R. Burridge and G. Papanicolaou, *The geometry of coupled mode propagation in one-dimensional random media*, Comm. Pure Appl. Math., 25 (1972), pp. 715–757.

[4] L. B. Dozier and F. D. Tappert, *Statistics of normal mode amplitudes in a random ocean* I & II, J. Acoust. Soc. Am., 63 (1978), pp. 353–365, 533–547.

[5] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna, *Wave Propagation and Time Reversal in Randomly Layered Media*, Springer-Verlag, New York, 2007.

[6] J. Garnier, *The role of evanescent modes in randomly perturbed single-mode waveguides*, Discrete Contin. Dyn. Syst. Ser. B, 8 (2007), pp. 455–472.

[7] J. Garnier and G. Papanicolaou, *Pulse propagation and time reversal in random waveguides*, SIAM J. Appl. Math., 67 (2007), pp. 1718–1739.

[8] W. Kohler and G. Papanicolaou, *Wave propagation in randomly inhomogeneous ocean*, in Wave Propagation and Underwater Acoustics, Lecture Notes in Phys. 70, J. B. Keller and J. S. Papadakis, eds., Springer-Verlag, Berlin, 1977, pp. 153–223.

[9] Y. Kuga and A. Ishimaru, *Retroreflectance from a dense distribution of spherical particles*, J. Opt. Soc. Amer., 1 (1984), pp. 831–835.

[10] H. Kushner, *Approximation and Weak Convergence Methods for Random Processes, with Applications to Stochastic Systems Theory*, MIT Press Ser. Signal Process. Optim. Control 6, MIT Press, Cambridge, MA, 1984.

[11] G. Labeyrie, F. de Tomasi, J.-C. Bernard, C. A. Müller, C. Miniatura, and R. Kaiser, *Coherent backscattering of light by atoms*, Phys. Rev. Lett., 83 (1999), pp. 5266–5269.

[12] G. Papanicolaou and S. Weinryb, *A functional limit theorem for waves reflected by a random medium*, Appl. Math. Optim., 30 (1994), pp. 307–334.

[13] H. E. Rowe, *Electromagnetic Propagation in Multi-Mode Random Media*, Wiley, New York, 1999.

[14] A. Tourin, A. Derode, P. Roux, B. A. van Tiggelen, and M. Fink, *Time-dependent coherent backscattering of acoustic waves*, Phys. Rev. Lett., 79 (1997), pp. 3637–3639.
[15] M. P. van Albada and A. Lagendijk, *Observation of weak localization of light in a random medium*, Phys. Rev. Lett., 55 (1985), pp. 2692–2695.
[16] M. C. W. van Rossum and Th. M. Nieuwenhuizen, *Multiple scattering of classical waves: Microscopy, mesoscopy, and diffusion*, Rev. Modern Phys., 71 (1999), pp. 313–371.
[17] P. E. Wolf and G. Maret, *Weak localization and coherent backscattering of photons in disordered media*, Phys. Rev. Lett., 55 (1985), pp. 2696–2699.
[18] K. M. Yoo, G. C. Tang, and R. R. Alfano, *Coherent backscattering of light from biological tissues*, Appl. Optics, 29 (1990), pp. 3237–3239.