Propagation of partially coherent light with the Maxwell–Debye equation

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We deal with the propagation of broadband Schell-model sources in nonlinear media with finite relaxation time. The approach is based on a study of the Wigner distribution function and on a separation of scales technique between the microscopic random fluctuations of the field and the macroscopic intensity profile. The regime in which the nonlinearity is strong and slow is considered. Precise results are obtained for the small- and large-scale characteristics of the pulse: optical intensity profile, speckle radius, and typical intensity profile of the speckle spots. © 2003 Optical Society of America

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1. INTRODUCTION

We aim to study the propagation of a broadband pulse with speckled intensity profile and to exhibit the main small- and large-scale features relative to this propagation. The literature contains much research devoted to the linear propagation of partially coherent beams,[1] among which the Schell model sources characterized by a complex degree of coherence between two points $\mathbf{r}_1$ and $\mathbf{r}_2$ in the source plane that depends only on the difference $\mathbf{r}_1 - \mathbf{r}_2$ have attracted attention. In particular Gaussian Schell-model (GSM) sources characterized by Gaussian distributions for both the optical intensity and the complex degree of spatial coherence have been extensively studied.[2,3] This analysis came about mainly because the GSM sources can be constructed in a laboratory[4] and are mathematically tractable to provide relevant insight into the phenomena at hand. Analytical results are restricted mainly to the GSM beams,[2] twisted GSM beams,[5] J_0-correlated sources,[6] and further incoherent superpositions of Gaussian modes.[7] These results can provide qualitative insight into the propagation of other kinds of partially coherent beam, but original behavior can be exhibited especially when dealing with super-Gaussian profiles.[8]

The literature with regard to the linear propagation of partially coherent light is extensive. There are however fewer papers that address nonlinear propagation. Propagation of the GSM beams in dispersive and absorbing media has been studied by Wang and Wolf.[9] The propagation of an intense partially coherent quasi-monochromatic source in a two-dimensional Kerr medium was discussed by Gross and Manasssah.[10] The case of the instantaneous nonlinear Schrödinger equation was addressed by Garnier *et al.*[11] It was proved that the main contribution of the instantaneous nonlinearity is to break the Gaussian property of the statistical distribution and to enhance the local intensity fluctuations. Nayyar considered the propagation of the autocorrelation function (AF) of a GSM beam in a medium with third-order nonlinearity,[12] in the regime of rapid and weak nonlinearity. Here we consider a different regime, in which the nonlinearity has a greater amplitude but is slower than the coherence time of the pulse, and we also consider sources with general profiles and not only GSM sources.

Our analysis is based on a separation of scales technique, also referred to in the literature as the quasi-homogeneous approximation.[1] We do not perform an expansion for small nonlinearity as we consider the general configurations in which diffraction and nonlinearity are of the same order, so that coupling between the natural diffractive spreading and the nonlinear self-focusing is efficient. We address the nonlinear regime within a general framework, when the degree of coherence and the overall envelope of the optical intensity have arbitrary profiles. Our main assumption is that the macroscopic radius of the input beam is much larger than the correlation radius of the microscopic random fluctuations of the field. In the case of a slow response of the material we prove that the Gaussian property of the statistical distribution is preserved along propagation (note that we do not assume it but we actually prove it in Section 4). Once this fact is established, it remains to derive an equation for the second-order moments that fully characterize the pulse statistics, in that we can get from the AF both the macroscopic variations of the optical intensity, as well as the statistical distribution of the random microscopic fluctuations of the field. The analysis performed in Section 5 shows that it is convenient to work with the Wigner function, which is the local Fourier transform of the AF of the field. Section 6 is devoted to the derivation of the nonlinear hyperbolic system that governs the evolution of the Wigner function. The integration of this partial differential equation provides answers for all the physically relevant questions. In Section 8 we see that the results strongly depend on the initial global and local shapes of the initial AF.

2. MAXWELL–DEBYE EQUATION

A light pulse can excite material oscillation modes with different physical origins. As a result they cover an enormous frequency bandwidth that ranges from 10^{15} s^{-1} (electronic oscillations) to 1 s^{-1} (thermal oscillations). The most common contribution to the optical nonlinearity is the nonresonant electronic Kerr effect, which is usually small and fast. But light can also couple to material waves by inducing dipoles in the atoms or the molecules making up the material. These phenomena are both stronger and slower than the electronic one. We might think of molecular reorientation ($\sim 10^{12}$ s^{-1}), electrostriction ($\sim 10^{9}$ s^{-1}), or else thermal nonlinearities ($\sim 1-10^{6}$ s^{-1}). In 1912 Debye was the first to address the problem of reorientation of polar molecules in an electric field.[13] As a result of this pioneering study the interaction of an electromagnetic wave in nonresonant media with finite response time τ_R is modeled by the coupled field–matter equations known as the Maxwell–Debye equation:[14]

$$i\frac{\partial E}{\partial z} + \frac{1}{2k_0}\Delta_\perp E + \frac{k_0 n_{\mathrm{nl}}}{n_0}E = 0, \tag{1}$$

$$\tau_R\frac{\partial n_{\mathrm{nl}}}{\partial t} + n_{\mathrm{nl}} = n_2|E|^2, \tag{2}$$

where z is the wave-propagation axis and $\Delta_\perp$ is the orthogonal Laplacian operator $(\partial^2/\partial x^2) + (\partial^2/\partial y^2)$ that describes the diffraction of the wave in the transverse plane. We denote by $\mathbf{r}$ the vector that consists of the transverse coordinates (x, y) and by t the local time in the moving pulse time frame. n_0 is the unperturbed value of the index of refraction and n_{nl} is the optical nonlinear coefficient that governs the response of the medium to the wave amplitude. τ_R is the relaxation time of the medium. Note that Eq. (2) can be solved in terms of the pulse intensity:

$$n_{\mathrm{nl}}(z, \mathbf{r}, t) = \frac{n_2}{\tau_R}\int_{-\infty}^{t}|E|^2(z, \mathbf{r}, s)\exp[(t-s)/\tau_R]\mathrm{d}s. \tag{3}$$

We study the propagation of partially coherent light whose statistical characteristics are locally stationary. More precisely, we consider an incident field E_0 whose temporal and spatial slowly varying envelopes are deterministic, with spatial radius r_0 (the so-called beam radius) and temporal duration t_0 (the so-called pulse duration). These scales are referred to in the following as the macroscopic scales. The incident pulse also has fast varying random fluctuations, with characteristic scales ρ_0 (the so-called correlation radius) in the spatial domain and τ_0 (the so-called coherence time) in the temporal domain. These scales are referred to in the following as the microscopic scales. The random fluctuations of the incident pulse are assumed to obey Gaussian statistics. In particular the spatial intensity distribution is a speckle pattern,[15] and the field is a Schell-model source.[1] The distribution of the input field is characterized in the $z = 0$ plane by the AF of field $E_0(\mathbf{r}, t) = E(z = 0, \mathbf{r}, t)$ defined by

$$C_{0,\mathbf{r},t}(\boldsymbol{\rho}, \tau) = \langle E_0(\mathbf{r}_1, t_1)E_0{}^*(\mathbf{r}_2, t_2)\rangle, \tag{4}$$

where $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, $\boldsymbol{\rho} = \mathbf{r}_2 - \mathbf{r}_1$, $t = (t_1 + t_2)/2$, $\tau = t_2 - t_1$, and the angular brackets represent statistical averaging. We can assume, for example, that the AF has a locally Gaussian shape, so that expression $C_{0,\mathbf{r},t}$ can be taken to be equal to

$$C_{0,\mathbf{r},t}(\rho, t) = I_0(\mathbf{r}, t)\exp\left(-\frac{|\boldsymbol{\rho}|^2}{\rho_0{}^2} - \frac{\tau^2}{\tau_0{}^2}\right). \tag{5}$$

$I_0 := I_0(0, 0)$ is the mean intensity of the field at the center of the beam. $\mathbf{r} \mapsto I_0(\mathbf{r}, t)$ and $t \mapsto I_0(\mathbf{r}, t)]$ describe, respectively, the spatial and temporal shapes of the slowly varying envelope of the intensity. For the numerical experiments we also assume that $I_0(\mathbf{r}, t)$ is of the form

$$I_0(\mathbf{r}, t) = I_0\exp\left(-\frac{|\mathbf{r}|^p}{r_0{}^p} - \frac{t^q}{t_0{}^q}\right), \tag{6}$$

where p and q are even integers. If p is equal to 2, the slowly varying envelope of the intensity has a Gaussian shape (this case corresponds to the so-called GSM source[2]), and if p is larger than 2 the envelope has a super-Gaussian shape.

On the one hand we prove that the Gaussian property of the statistics of the field is conserved by Eqs. (1) and (2) within the framework we consider. We then focus on the evolution of the four-dimensional AF:

$$C_{z,\mathbf{r},t}(\zeta, \boldsymbol{\rho}, \tau) := \langle E(z_1, \mathbf{r}_1, t_1)E^*(z_2, \mathbf{r}_2, t_1)\rangle, \tag{7}$$

where $z = (z_1 + z_2)/2$, $\zeta = z_2 - z_1$, $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, $\boldsymbol{\rho} = \mathbf{r}_2 - \mathbf{r}_1$, $t = (t_1 + t_2)/2$, and $\tau = t_2 - t_1$. This function is particularly relevant because it provides the beam radius and the pulse duration as well as information about the statistical distribution of the microscopic hot spots.

3. SEPARATION OF THE TIME SCALES

To obtain an accurate description of the evolution of the statistics of the hot spots, we intend to study the evolution of the autocorrelation function. It does not seem possible to get a closed-form equation that governs the evolution of the AF, because nonlinearity makes the nth-order correlation depend on the $(n + 2)$th-order correlation. Several authors considered the case when the characteristic distance over which the nonlinear interaction between the field and the fluctuations of $k_0 n_{\mathrm{nl}}(E)/n_0$ takes place is much greater than the size of the region of longitudinal field correlation $\zeta_0 = k_0\rho_0{}^2$, which is determined by the length of diffraction spreading of a characteristic inhomogeneity in the cross section of an incoherent beam.[12,16,17] Under such conditions the statistics are only slightly affected by the nonlinear effect, so that the fourth-order moments can be deduced from the second-order moments according to the simple rules that are valid for Gaussian processes[18] and a closed-form equation for the AF can be obtained. Our framework is different because we take into consideration the slow nonlinear mechanism whose strength is much higher than the almost-instantaneous Kerr effect.

The first hypothesis is that coherence time τ_0 of the input pulse is shorter than relaxation time τ_R of the medium. As a consequence the medium averages out the fast oscillations by Eq. (3). The nonlinear index $n_{\rm nl}$ depends only on the locally time-averaged intensity, which is equal by ergodicity to the local statistical average of the intensity. We thus have

$$n_{\rm nl}(z, \mathbf{r}, t) = \frac{n_2}{\tau_R} \int_{-\infty}^{t} \langle |E|^2 \rangle (z, \mathbf{r}, s) \exp[(t - s)/\tau_R] \mathrm{d}s.$$

Furthermore, assuming that pulse duration t_0 is longer than τ_R, we have $\langle |E|^2 \rangle (z, \mathbf{r}, s) \simeq \langle |E|^2 \rangle (z, \mathbf{r}, t)$ for $|t - s| \lesssim \tau_R$ and the nonlinear index of refraction has the form

$$n_{\rm nl}(z, \mathbf{r}, t) = n_2 \langle |E|^2 \rangle (z, \mathbf{r}, t), \tag{8}$$

so that the field obeys the partial differential equation

$$i \frac{\partial E}{\partial z} + \frac{1}{2k_0} \Delta_\perp E + \frac{k_0 n_2}{n_0} \langle |E|^2 \rangle E = 0, \tag{9}$$

which simply corresponds to original Eqs. (1)–(3), where the statistical average was substituted for the time average.

4. GAUSSIAN PROPERTY

We now prove that, if $E_0(\cdot)$ has Gaussian statistics, then $E(z, \cdot)$ also has Gaussian statistics. Note that this is not obvious as the propagation is nonlinear and nonlinearity usually breaks the Gaussian property. As an example, for the instantaneous nonlinear Schrödinger equation, where $n_{\rm nl} = n_2 |E|^2$, Garnier *et al.* proved[11] that the main effect of the nonlinear propagation was to destroy the Gaussian property of the statistics by the addition of a nonlinear phase $\phi_{\rm NL} = k_0 n_2 |E_0|^2 z / n_0$, whereas the main effect on the intensity distribution was a contrast enhancement that showed that the fourth-order correlation function differed from the standard sum of products of second-order correlation functions that correspond to a Gaussian process.

We give complete proof of the conservation of the Gaussian property as we believe it is important to show that this is a result and not an assumption. The proof is based on two well-known propositions of the mathematical probability theory.

(1) The characterization of Gaussian processes: The statistical distribution of a zero-mean Gaussian process if fully characterized by an AF.

(2) The central-limit theorem: If X_j, $j = 1,\ldots, N$ are independent zero-mean processes (not necessarily Gaussian) with the same AF, C, then the normalized sum $1/\sqrt{N}\Sigma_{j=1}^{N} X_j$ converges as $N \to \infty$ to a zero-mean Gaussian process with AF C.

Consider N statistically independent versions of the input field $E_{j,0}$, $j = 1,\ldots, N$. We also introduce the sum

$$\tilde{E}_{N,0} := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} E_{j,0},$$

where $\tilde{E}_{N,0}$ is the sum of independent Gaussian processes, so that it is a Gaussian process. Furthermore its AF is C_0. Thus its statistical distribution is the same as E_0. We denote by E_j the solution of evolution Eq. (9) starting from $E_{j,0}$. The E_j are statistically independent and identically distributed, but we still do not know whether they are Gaussian processes. Let us introduce

$$\tilde{E}_N := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} E_j. \tag{10}$$

The equation that governs the evolution of $\tilde{E}_N$ is

$$i \frac{\partial \tilde{E}_N}{\partial z} + \frac{1}{2k_0} \Delta_\perp \tilde{E}_N + \frac{k_0 n_2}{n_0} \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \langle |E_j|^2 \rangle E_j = 0.$$

However the E_j are identically distributed, so that

$$\langle |E_j|^2 \rangle = \langle |E_1|^2 \rangle, \quad j = 1,\ldots, N. \tag{11}$$

Furthermore

$$\langle |\tilde{E}_N|^2 \rangle = \frac{1}{N} \sum_{j,l=1}^{N} \langle E_j E_l^* \rangle.$$

Since the E_j terms are statistically independent we have

$$\langle |\tilde{E}_N|^2 \rangle = \frac{1}{N} \sum_{j \neq l} \langle E_j \rangle \langle E_l^* \rangle + \frac{1}{N} \sum_{j=1}^{N} \langle |E_j|^2 \rangle.$$

The first sum on the right-hand side is zero as the E_j are zero-mean processes, whereas the second sum is equal to $\langle |E_1|^2 \rangle$ by Eq. (11). Accordingly, the equation that governs the evolution of $\tilde{E}_N$ is

$$i \frac{\partial \tilde{E}_N}{\partial z} + \frac{1}{2k_0} \Delta_\perp \tilde{E}_N + \frac{k_0 n_2}{n_0} \langle |\tilde{E}_N|^2 \rangle \tilde{E}_N = 0.$$

In summary, $\tilde{E}_N$ is a solution of evolution Eq. (9) and it starts from $\tilde{E}_{N,0}$ that has the same statistical distribution as E_0. Thus E and $\tilde{E}_N$ have the same statistical distribution whatever z is. Note that this also holds true whatever N is. Since $\tilde{E}_N$ is defined as the sum of Eq. (10) of independent processes, the central-limit theorem can be invoked to prove that as $N \to \infty$ $\tilde{E}_N$ converges to a process with Gaussian statistics. This definitively proves that E has Gaussian statistics.

The Gaussian property of the field that we have just put into evidence allows us to claim several interesting statements. First the local contrast of the intensity distribution is 1 for every z (the contrast is defined as the ratio of the standard deviation of the local fluctuations over the mean intensity). This means that there is no filamentation. Furthermore the four-dimensional AF as defined by Eq. (7) contains all the relevant information. In particular, for a fixed time t the three-dimensional (3-D) AF

$$\gamma^{3\text{-D}}(z, \mathbf{r}; \zeta, \boldsymbol{\rho}) := C_{z,\mathbf{r},t}(\zeta, \boldsymbol{\rho}, 0) = \left\langle E\left(z + \frac{\zeta}{2}, \mathbf{r} + \frac{\boldsymbol{\rho}}{2}, t\right) E^*\left(z - \frac{\zeta}{2}, \mathbf{r} - \frac{\boldsymbol{\rho}}{2}, t\right)\right\rangle \quad (12)$$

contains all the information about the small- and large-scale spatial structures of the beam. In particular it gives the global radius of the optical intensity profile since $\mathbf{r} \mapsto \gamma^{3\text{-D}}(z, \mathbf{r}; 0, \mathbf{0})$ is precisely the overall envelope of the optical intensity profile. This function also describes the typical 3-D shapes of the speckle spots of the beam. Indeed well-known results about Gaussian processes[19] have established that the typical behavior of the field around a local maximum located *in situ* $(Z, \mathbf{R})$ is precisely given by the function $(\zeta, \boldsymbol{\rho}) \mapsto \gamma^{3\text{-D}}(Z, \mathbf{R}; \zeta, \boldsymbol{\rho})$:

$$|E(Z + \zeta, \mathbf{R} + \boldsymbol{\rho})| \simeq |E(Z, \mathbf{R})| \frac{|\gamma^{3\text{-D}}(Z, \mathbf{R}; \zeta, \boldsymbol{\rho})|}{|\gamma^{3\text{-D}}(Z, \mathbf{R}; 0, \mathbf{0})|}. \quad (13)$$

This result holds true for values of $(\zeta, \boldsymbol{\rho})$ small enough so that the right-hand side is larger than $\sqrt{I_0}$.

From now on we assume that the input AF is of the form

$$C_{0,\mathbf{r},t}(\boldsymbol{\rho}, \tau) = f_t(\tau)\gamma_0(\mathbf{r}, \boldsymbol{\rho}).$$

Substituting Eq. (7) into Eq. (9) immediately yields a four-dimensional AF of the form

$$C_{z,\mathbf{r},t}(\zeta, \boldsymbol{\rho}, \tau) = f_t(\tau)\gamma^{3\text{-D}}(z, \mathbf{r}; \zeta, \boldsymbol{\rho}),$$

where $\gamma^{3\text{-D}}$ is the 3-D AF. This shows that the coherence time and pulse duration are not affected by the propagation, as time does not appear explicitly in Eq. (9). Accordingly we can now focus our attention on the spatial AF. We first study the transverse AF in Section 5 before we address the 3-D AF.

5. WIGNER FORMALISM

We consider the solution of Eq. (9) starting from $E(z = 0, \mathbf{r}) = E_0(\mathbf{r})$, which is a field with Gaussian statistics characterized by the Wigner function[20]

$$W_0(\mathbf{r}, \mathbf{k}) = \frac{1}{(2\pi)^2} \int \exp(i\mathbf{k}\cdot\boldsymbol{\rho})\gamma_0(\mathbf{r}, \boldsymbol{\rho})\mathrm{d}\boldsymbol{\rho}, \quad (14)$$

where γ_0 is the AF,

$$\gamma_0(\mathbf{r}, \boldsymbol{\rho}) = \left\langle E_0\left(\mathbf{r} - \frac{\boldsymbol{\rho}}{2}\right) E_0^*\left(\mathbf{r} + \frac{\boldsymbol{\rho}}{2}\right)\right\rangle. \quad (15)$$

We look for a closed equation for the Wigner function

$$W(z, \mathbf{r}, \mathbf{k}) = \frac{1}{(2\pi)^2} \int \exp(i\mathbf{k}\cdot\boldsymbol{\rho})\gamma(z, \mathbf{r}, \boldsymbol{\rho})\mathrm{d}\boldsymbol{\rho}, \quad (16)$$

where

$$\gamma(z, \mathbf{r}, \boldsymbol{\rho}) = \left\langle E\left(z, \mathbf{r} - \frac{\boldsymbol{\rho}}{2}\right) E^*\left(z, \mathbf{r} + \frac{\boldsymbol{\rho}}{2}\right)\right\rangle. \quad (17)$$

Note that we have

$$\langle |E(z, \mathbf{r})|^2\rangle = \int W(z, \mathbf{r}, \mathbf{k})\mathrm{d}\mathbf{k}. \quad (18)$$

Straightforward algebra establishes that the evolution of W is governed by the equation

$$\frac{\partial W}{\partial z} + \frac{\mathbf{k}}{k_0} \cdot \nabla_r W + \frac{k_0 n_2}{n_0}\mathcal{L}W = 0. \quad (19)$$

Operator $\mathcal{L}$ is defined by

$$\mathcal{L}Z(\mathbf{r}, \mathbf{k}) = -i \int \exp(-i\mathbf{p}\cdot\mathbf{r})\hat{V}(\mathbf{p}) \times \left[Z\left(\mathbf{r}, \mathbf{k} + \frac{\mathbf{p}}{2}\right) - Z\left(\mathbf{r}, \mathbf{k} - \frac{\mathbf{p}}{2}\right)\right]\mathrm{d}\mathbf{p}, \quad (20)$$

where $\hat{V}$ is the Fourier transform with respect to $\mathbf{r}$ of

$$V(z, \mathbf{r}) := \langle |E|^2\rangle(z, \mathbf{r}) = \int W(z, \mathbf{r}, \mathbf{k})\mathrm{d}\mathbf{k}.$$

Let us write the input AF in the form

$$\gamma_0(\mathbf{r}, \boldsymbol{\rho}) = I_0\bar{\gamma}_0\left(\frac{\mathbf{r}}{r_0}, \frac{\boldsymbol{\rho}}{\rho_0}\right),$$

where r_0 is the radius of the intensity envelope, ρ_0 is the correlation radius, and I_0 is the mean intensity at the beam center so that $\bar{\gamma}_0(\mathbf{0}, \mathbf{0}) = 1$. We consider the following dimensionless version of the Wigner function:

$$W(z, \mathbf{r}, \mathbf{k}) = I_0{\rho_0}^2\bar{W}\left(\frac{z}{z_0}, \frac{\mathbf{r}}{r_0}, \rho_0\mathbf{k}\right),$$

so that

$$\mathbf{k} \cdot \nabla_r W(z, \mathbf{r}, \mathbf{k}) = \frac{I_0{\rho_0}^2}{\rho_0 r_0}[\bar{\mathbf{k}} \cdot \nabla_{\bar{r}}\bar{W}(\bar{z}, \bar{\mathbf{r}}, \bar{\mathbf{k}})]_{\bar{z}=z/z_0, \bar{r}=r/r_0, \bar{\mathbf{k}}=\mathbf{k}\rho_0},$$

$$\frac{\partial W}{\partial z}(z, \mathbf{r}, \mathbf{k}) = \frac{I_0{\rho_0}^2}{z_0}\left[\frac{\partial \bar{W}}{\partial \bar{z}}(\bar{z}, \bar{\mathbf{r}}, \bar{\mathbf{k}})\right]_{\bar{z}=z/z_0, \bar{r}=r/r_0, \bar{\mathbf{k}}=\mathbf{k}\rho_0},$$

$$V(z, \mathbf{r}) = I_0[\bar{V}(\bar{z}, \bar{\mathbf{r}})]_{\bar{z}=z/z_0, \bar{\mathbf{r}}=r/r_0},$$

$$\bar{V}(\bar{z}, \mathbf{r}) = \int \bar{W}(\bar{z}, \bar{\mathbf{r}}, \bar{\mathbf{k}})d\bar{\mathbf{k}},$$

$$\hat{V}(z, \mathbf{p}) = I_0{r_0}^2[\hat{\bar{V}}(\bar{z}, \bar{\mathbf{p}})]_{\bar{z}=z/z_0\bar{p}=\mathbf{p}r_0},$$

$$\mathcal{L}W(z, \mathbf{r}, \mathbf{k}) = {I_0}^2{\rho_0}^2[\bar{\mathcal{L}}\bar{W}(\bar{z}, \bar{\mathbf{r}}, \bar{\mathbf{k}})]_{\bar{z}=z/z_0, \bar{\mathbf{r}}=r/r_0, \bar{\mathbf{k}}=\mathbf{k}\rho_0},$$

where the rescaled operator $\bar{\mathcal{L}}$ is defined by

$$\bar{\mathcal{L}}Z(\bar{\mathbf{r}}, \bar{\mathbf{k}}) = -i \int \exp(-i\bar{\mathbf{p}}\cdot\bar{\mathbf{r}})\hat{\bar{V}}(\bar{\mathbf{p}})\left[Z\left(\bar{\mathbf{r}}, \bar{\mathbf{k}} + \frac{\rho_0}{r_0}\frac{\bar{\mathbf{p}}}{2}\right) - Z\left(\bar{\mathbf{r}}, \bar{\mathbf{k}} - \frac{\rho_0}{r_0}\frac{\bar{\mathbf{p}}}{2}\right)\right]\mathrm{d}\bar{\mathbf{p}},$$

and $\hat{\bar{V}}$ is the Fourier transform of $\bar{V}$. For the characteristic propagation distance we chose

$$z_0 = \frac{k_0 r_0 \rho_0}{2}, \tag{21}$$

that is, the length of diffraction spreading of a partially coherent beam with macroscopic radius r_0 and microscopic correlation radius ρ_0.[1,8] Finally, we denote

$$\bar{\beta} = \frac{n_2 I_0 k_0^2 \rho_0^2}{2 n_0},$$

so that the equation that governs the evolution of the dimensionless Wigner function is

$$\frac{\partial \bar{W}}{\partial \bar{z}} + \frac{1}{2}\bar{\mathbf{k}} \cdot \nabla_{\bar{\mathbf{r}}} \bar{W} + \bar{\beta}\frac{r_0}{\rho_0}\bar{\mathcal{L}}\bar{W} = 0. \tag{22}$$

Note that $\bar{\beta}$ can be written in terms of critical power $P_c = n_0^2 c/(4 n_2 k_0^2)$ and the typical power contained in a speckle spot $P_0 = n_0 c I_0 \rho_0^2/8$ as

$$\bar{\beta} = P_0/P_c. \tag{23}$$

6. SEPARATION OF THE SPATIAL SCALES

We shall now use the second scaling hypothesis that the typical length scale of the random fluctuations of the field is much smaller than the radius of the beam envelope: $r_0 \gg \rho_0$. In such conditions operator $\bar{\mathcal{L}}$ can be simplified to

$$\bar{\mathcal{L}}Z(\bar{\mathbf{r}}, \bar{\mathbf{k}}) = -i\frac{\rho_0}{r_0}\int \exp(-i\bar{\mathbf{p}}\cdot\bar{\mathbf{r}})\hat{\bar{V}}(\bar{\mathbf{p}})[\nabla_{\bar{k}}Z(\bar{\mathbf{r}}, \bar{\mathbf{k}}) \cdot \bar{\mathbf{p}}]\mathrm{d}\bar{\mathbf{p}}$$

so that integration by parts yields

$$\bar{\mathcal{L}}Z(\bar{\mathbf{r}}, \bar{\mathbf{k}}) = \frac{\rho_0}{r_0}\nabla_{\bar{r}}\bar{V} \cdot \nabla_{\bar{k}}Z(\bar{\mathbf{r}}, \bar{\mathbf{k}}). \tag{24}$$

Substituting Eq. (24) into Eq. (22) we determine that the equation that governs the evolution of the Wigner function is a Liouville–Vlasov type in phase space:

$$\frac{\partial \bar{W}}{\partial \bar{z}} + \frac{1}{2}\bar{\mathbf{k}} \cdot \nabla_{\bar{r}}\bar{W} + \bar{\beta}\nabla_{\bar{\mathbf{r}}}\bar{V} \cdot \nabla_{\bar{\mathbf{k}}}\bar{W} = 0, \tag{25}$$

where $\bar{V}(\bar{z}, \bar{\mathbf{r}}) = \int \bar{W}(\bar{z}, \bar{\mathbf{r}}, \bar{\mathbf{k}})\mathrm{d}\bar{\mathbf{k}}$.

Equation (25) is the most important result of this paper. It gives a simple and efficient way to predict the behavior of a partially coherent pulse in a slow nonlinear medium. It is a nonlinear hyperbolic system that is likely to involve shock formation. In Section 7 we see that this is indeed the case if the nonlinearity is strong enough. The reader would be surprised to learn that such a singularity arises because we have pointed out that there is no possible singularity in Eq. (9). The reason is that a shock within the framework of the separation of scales simply means that the global envelope has a sharp edge whose length scale has the same order of magnitude as the microscopic random fluctuations. Accordingly, Eq. (25) accurately describes the nonlinear propagation although there is no shock. When the shock arises, our approach fails because the separation of spatial scales technique is no longer valid, and one should compute the propagation of the Wigner function with Eq. (9).

Note that dimensional analysis puts into evidence that the interesting nonlinear regime occurs when $\bar{\beta}$ is of order one, that is to say when the mean power of a speckle spot is of the order of the critical power. Within this framework the diffraction effects and the nonlinearity are of the same magnitude. If the nonlinearity is much weaker, it can be neglected and the analysis performed in the linear regime is valid.[8] If the nonlinearity is much stronger, the diffractive effect can be neglected and the main effect is a spectral broadening induced by self-phase modulation.

7. THREE-DIMENSIONAL AUTOCORRELATION FUNCTION

The above analysis gives us access to the transverse AF through a Fourier transform

$$\gamma(z, \mathbf{r}, \boldsymbol{\rho}) = \int W(z, \mathbf{r}, \mathbf{k})\exp(-i\mathbf{k}\cdot\boldsymbol{\rho})\mathrm{d}\mathbf{k}.$$

We can then get the full 3-D AF as defined by Eq. (12) by solving the equation

$$i\frac{\partial \gamma^{3\text{-D}}}{\partial \zeta} + \frac{1}{2k_0}\Delta_{\boldsymbol{\rho}}\gamma^{3\text{-D}} + \frac{k_0 n_2}{n_0}V(z, \mathbf{r})\gamma^{3\text{-D}} = 0, \tag{26}$$

starting from $\gamma^{3\text{-D}}(z, \mathbf{r}; \zeta = 0, \boldsymbol{\rho}) = \gamma(z, \mathbf{r}, \boldsymbol{\rho})$. Note that the macroscopic variables z and $\mathbf{r}$ play the roles of frozen parameters in this equation, which reads as a Schrödinger-like equation in the constant potential V. Let us consider the 3-D Wigner transform

$$W^{3\text{-D}}(z, \mathbf{r}; \kappa, \mathbf{k}) := \frac{1}{(2\pi)^3}\iint \exp(i\mathbf{k}\cdot\boldsymbol{\rho} + i\kappa\zeta)\gamma^{3\text{-D}}(z, \mathbf{r}; \zeta, \boldsymbol{\rho})\mathrm{d}\boldsymbol{\rho}\mathrm{d}\zeta.$$

In the phase space Eq. (26) reads as

$$\left[\kappa - \frac{1}{2k_0}|\mathbf{k}|^2 + \frac{k_0 n_2}{n_0}V(z, \mathbf{r})\right]W^{3\text{-D}} = 0,$$

$$\int W^{3\text{-D}}(z, \mathbf{r}; \kappa, \mathbf{k})\mathrm{d}\kappa = W(z, \mathbf{r}, \mathbf{k}),$$

whose solution is

$$W^{3\text{-D}}(z, \mathbf{r}; \kappa, \mathbf{k}) = W(z, \mathbf{r}, \mathbf{k})\delta[\kappa - K(z, \mathbf{r}, \mathbf{k})], \tag{27}$$

$$K(z, \mathbf{r}, \mathbf{k}) = \frac{1}{2k_0}|\mathbf{k}|^2 - \frac{k_0 n_2}{n_0}V(z, \mathbf{r}), \tag{28}$$

where δ is the Dirac distribution. Taking the inverse Fourier transform establishes

$$\gamma^{3\text{-D}}(z, \mathbf{r}; \zeta, \boldsymbol{\rho}) = \int \mathrm{d}\mathbf{k}W(z, \mathbf{r}, \mathbf{k}) \times \exp[-i\mathbf{k}\cdot\boldsymbol{\rho} - iK(z, \mathbf{r}, \mathbf{k})\zeta]. \tag{29}$$

In summary, we have proved that the statistical distribution of the partially coherent beam is Gaussian. It has zero mean and its 3-D AF is given by Eq. (29), where W is the solution of the nonlinear partial differential Eq. (25). We have thus completely characterized the small- and large-scale behavior of the pulse.

8. NUMERICAL INTEGRATION

Here we present numerical experiments of the nonlinear partial differential Eq. (25). For simplicity and good resolution we assume circular symmetry for the statistical distribution. This is valid as soon as the initial Wigner distribution $\bar{W}_0(\bar{\mathbf{r}}, \bar{\mathbf{k}})$ depends only on the moduli of $\bar{\mathbf{k}}$ and $\bar{\mathbf{r}}$ and on the angle between $\bar{\mathbf{k}}$ and $\bar{\mathbf{r}}$. In such conditions the Wigner function $\bar{W}(\bar{z},.)$ depends only on three variables instead of four, that is to say, $\bar{r} = |\bar{\mathbf{r}}|$, $\bar{k} = |\bar{\mathbf{k}}|$, and θ is the angle between $\bar{\mathbf{r}}$ and $\bar{\mathbf{k}}$: $\cos(\theta) = (\bar{\mathbf{r}} \cdot \bar{\mathbf{k}})/(|\bar{\mathbf{r}}||\bar{\mathbf{k}}|)$. In these variables Eq. (25) reads

$$\frac{\partial \bar{W}}{\partial \bar{z}} + \frac{1}{2}\bar{k}\left[\cos(\theta)\frac{\partial \bar{W}}{\partial \bar{r}} - \frac{1}{\bar{r}}\sin(\theta)\frac{\partial \bar{W}}{\partial \theta}\right] + \bar{\beta}\frac{\partial \bar{V}}{\partial \bar{r}}\left[\cos(\theta)\frac{\partial \bar{W}}{\partial \bar{k}} - \frac{1}{\bar{k}}\sin(\theta)\frac{\partial \bar{W}}{\partial \theta}\right] = 0.$$

Let us introduce the variable $u = \cos(\theta)$. We then get a conservative form for the nonlinear partial differential equation

$$\frac{\partial \bar{W}}{\partial \bar{z}} + \frac{\partial}{\partial \bar{r}}\left(\frac{u\bar{k}}{2}\bar{W}\right) + \frac{\partial}{\partial \bar{k}}\left(\bar{\beta}u\frac{\partial \bar{V}}{\partial \bar{r}}\bar{W}\right) + \frac{\partial}{\partial u}\left[(1-u^2)\left(\frac{\bar{k}}{2\bar{r}} + \frac{\bar{\beta}}{\bar{k}}\frac{\partial \bar{V}}{\partial \bar{r}}\right)\bar{W}\right] = 0.$$

We applied a second-order upwind Van Leer scheme[21] for the numerical resolution. In Appendix A we provide more details about the numerical method that we used.

Figures 1 and 2 plot the evolution of the intensity profiles $r \mapsto \gamma_z(r, 0)/I_0$ of partially coherent pulses for different positions along propagation axis z and for different input powers. We first consider a GSM source

$$\gamma_0(\mathbf{r}, \boldsymbol{\rho}) = I_0 \exp\left(-\frac{|\mathbf{r}|^2}{{r_0}^2} - \frac{|\boldsymbol{\rho}|^2}{{\rho_0}^2}\right),$$

which corresponds to the dimensionless Wigner function

$$\bar{W}_0(\bar{\mathbf{r}}, \bar{\mathbf{k}}) = \frac{1}{4\pi}\exp\left(-|\bar{\mathbf{r}}|^2 - \frac{|\bar{\mathbf{k}}|^2}{4}\right).$$

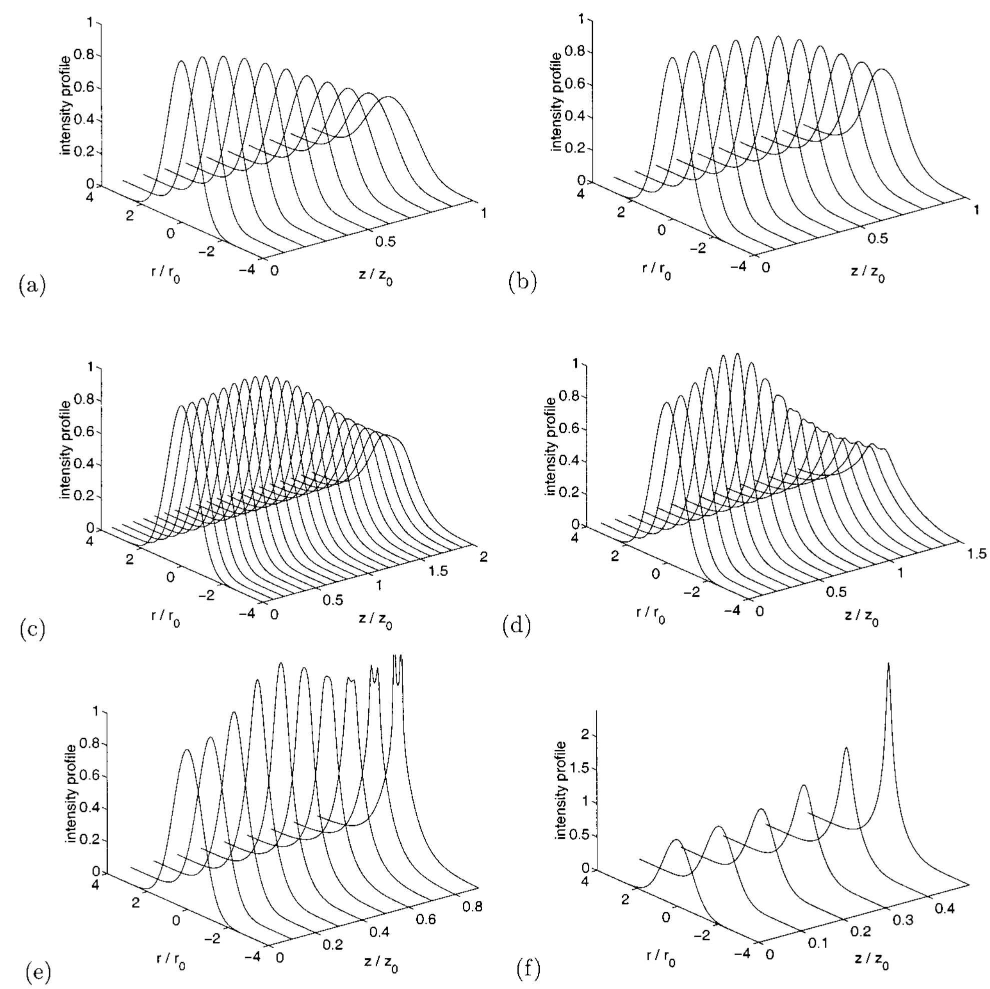

Fig. 1. Intensity profiles of the partially coherent beams in dimensionless variables. The initial profile has a Gaussian shape: (a) $\bar{\beta} = 0$ (linear regime), (b)–(f) $\bar{\beta} = 0.70, 0.98, 1.25, 1.41, 2.11$, respectively.

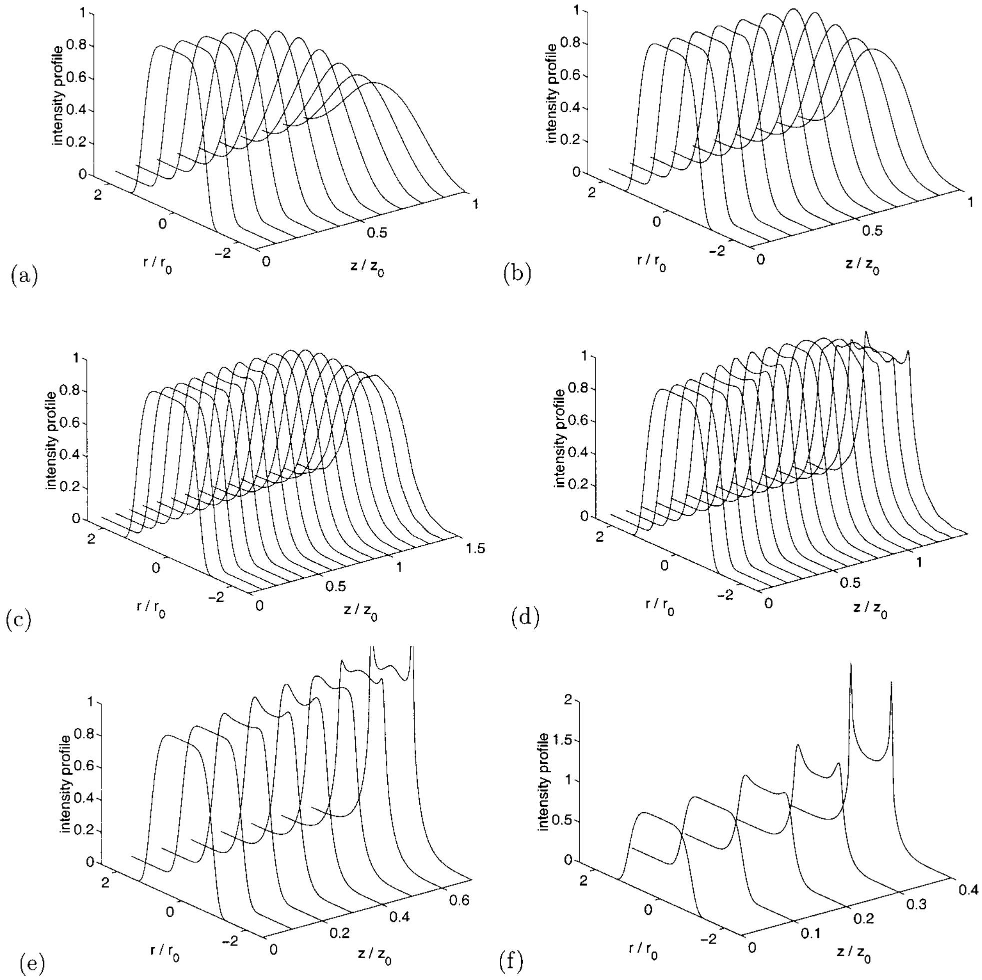

Fig. 2. Same as Fig. 1 but the initial profile has a super-Gaussian shape with order $2n = 6$.

The accuracy of the numerical scheme was first checked by a comparison of the numerical results in the linear regime (or equivalently for weak nonlinearity) with the closed-form expressions that are available[8] [see Fig. 1(a)]. We also consider nonlinear regimes in which self-focusing of the beam is noticeable [see Figs. 1(b)–1(f)]. A weak nonlinearity $\bar{\beta} = P_0/P_c < 1$ involves a delayed spreading of the beam profile. Note that a moderate nonlinearity could help the beam to maintain its shape for quite a long propagation distance [Figs. 1(b) and 1(c)]. However, it will eventually either spread out or collapse. Note also that it could happen [Fig. 1(d)] that the beam first focuses before spreading out. It might also experience a complicated transitory regime before it collapses [Fig. 1(e)]. A strong nonlinearity $\bar{\beta} > 1.5$ [Fig. 1(d)] is more dramatic in that the beam self-focuses and a singularity appears (in the sense discussed in Section 6).

We also address the case of a super-GSM, whose input AF is

$$\gamma_0(\mathbf{r},\ \boldsymbol{\rho}) = I_0 \exp\left(-\frac{|\mathbf{r}|^6}{{r_0}^6} - \frac{|\boldsymbol{\rho}|^2}{{\rho_0}^2} \right).$$

In the absence of nonlinearity we put into evidence the previously known result that the super-Gaussian shape is not preserved along the propagation [Fig. 2(a)]. We first observed that in case of weak nonlinearity, the intensity increases slightly at the beam center whereas the profile becomes Gaussian and finally the beam spreads out as the GSM source [Fig. 2(b)]. When $\bar{\beta} \simeq 1$, the beam maintains a super-Gaussian shape for quite a long propagation distance before spreading out [Fig. 2(c)]. In the case of a strong nonlinearity, the picture is quite different from the GSM case. In fact, it is obvious that the profile edges become sharp and that a focusing ring arises. The ring formation could take a while for moderate nonlinearity [Figs. 2(d) and 2(e)] whereas it is rapid for a strong nonlinearity [Fig. 2(f)]. This mechanism can be explained by some heuristic arguments. The medium is sensitive only to the macroscopic intensity profile. The nonlinearity involves the generation of a waveguide whose shape is imposed by this profile. Within this waveguide the speckle spots follow the gradient of the induced index of refraction. They have no reason to be affected at the beam center as the gradient of the index is close to zero, but they are pushed along the flanks of the waveguide to the area that corresponds to the index maximum. Accordingly the macroscopic intensity profile that results from the superposition of all the speckle spots becomes higher on the ring.

9. CONCLUSION

We have addressed the problem of the propagation of a partially coherent pulse in a nonlinear medium with slow nonlinearity. We have shown that the Gaussian property of the statistical distribution of the field is preserved if the coherence time of the pulse is shorter than the relaxation time of the medium. We have exhibited a closed-form equation that governs the evolution of the Wigner function that reads as a nonlinear hyperbolic system in phase space. We have shown that the linear diffraction and the nonlinear phenomena are in competition when the mean power of a speckle spot is of the order of the critical power of the medium. A moderate nonlinearity can help to maintain the initial intensity profile over one diffraction length or more. This regime is only transitory and is followed by a beam spreading or a beam collapse. In the case of strong nonlinearity we have shown various self-focusing processes that depend on the initial intensity profiles. In the particular case of a super-Gaussian intensity profile, we observed the apparition of a focusing ring.

APPENDIX A: NUMERICAL SCHEME FOR A NONLINEAR HYPERBOLIC SYSTEM

We detail the numerical scheme that we used to integrate nonlinear partial differential Eq. (25). For the sake of simplicity we drop the overbars. The following second-order upwind Van Leer scheme is taken from the survey by Leveque.[21]

1. Grids

Grid r: $r_{i+1/2}$, $i = 0,\ldots, I$ with uniform step δr; grid k: $k_{j+1/2}$, $i = 0,\ldots, J$ with uniform step δk; grid u: $u_{l+1/2}$, $l = 0,\ldots, L$ with uniform step δu; dual grid r:$r_i = 1/2(r_{i+1/2} + r_{i-1/2})$; dual grid k: $k_j = 1/2(k_{j+1/2} + k_{j-1/2})$; dual grid u: $u_l = 1/2(u_{l+1/2} + u_{l-1/2})$; approximation of W at point (r_i, k_j, u_l): $W_{i,j,l}$; approximation of W at $(r_{i+1/2}, k_j, u_l)$: $W_{i+1/2,j,l}$; approximation of W at $(r_i, k_{j+1/2}, u_l)$: $W_{i,j+1/2,l}$; and approximation of W at $(r_i, k_j, u_{l+1/2})$: $W_{i,j,l+1/2}$.

2. Discrete Evolution Equation

We implement a second-order conservative approximation for Eq. (25):

$$\frac{\partial W_{i,j,l}}{\delta z} + \frac{u_l k_j}{2\,\delta r}(W_{i+1/2,j,l} - W_{i-1/2,j,l}) + \frac{\beta u_l}{\delta k}\frac{\partial V_i}{\partial r}(W_{i,j+1/2,l} - W_{i,j-1/2,l}) + \frac{1 - {u_l}^2}{\delta u} \times \left(\frac{k_j}{2r_i} + \frac{\beta}{k_j}\frac{\partial V_i}{\partial r}\right)(W_{i,j,l+1/2} - W_{i,j,l-1/2}) = 0.$$

We used a second-order explicit Runge–Kutta method for the z discretization of this equation.

3. Evaluation of Flux $W_{i+1/2,j,l}$

$$W_{i+1/2,j,l} = W_{i,j,l} + \frac{\delta r}{2}{\sigma^r}_i \quad \text{if} \quad u_l > g0,$$

$$W_{i+1/2,j,l} = W_{i+1,j,l} - \frac{\delta r}{2}{\sigma^r}_{i+1} \quad \text{if} \quad u_l < 0,$$

where ${\sigma^r}_i$ is the limited slope

$${\sigma^r}_i = \frac{1}{\delta r}(W_{i+1,j,l} - W_{i,j,l})\phi(\theta^r),$$

$$\theta^r = \frac{W_{i,j,l} - W_{i-1,j,l}}{W_{i+1,j,l} - W_{i,j,l}},$$

$$\phi(\theta) = \frac{\theta + |\theta|}{1 + |\theta|}.$$

4. Evaluation of Flux $W_{i,j+1/2,l}$

$$W_{i,j+1/2,l} = W_{i,j,l} + \frac{\delta k}{2}{\sigma^k}_j \quad \text{if} \quad \beta u_l\frac{\partial V_i}{\partial r} > 0,$$

$$W_{i,j+1/2,l} = W_{i,j+1,l} - \frac{\delta k}{2}{\sigma^k}_{j+1} \quad \text{if} \quad \beta u_l\frac{\partial V_i}{\partial r} < 0,$$

where ${\sigma^k}_j$ is defined as for the r flux.

5. Evaluation of Flux $W_{i,j+1/2,l}$

$$W_{i,j,l+1/2} = W_{i,j,l} + \frac{\delta u}{2}{\sigma^u}_l \quad \text{if} \quad \frac{k_j}{2r_i} + \frac{\beta}{k_j}\frac{\partial V_i}{\partial r} > 0,$$

$$W_{i,j,l+1/2} = W_{i,j,l+1} - \frac{\delta u}{2}{\sigma^u}_{l+1} \quad \text{if} \quad \frac{k_j}{2r_i} + \frac{\beta}{k_j}\frac{\partial V_i}{\partial r} < 0,$$

where ${\sigma^u}_l$ is defined as for the r flux.

6. Evaluation of Gradient $\partial V_i / \partial r$

We use the identity $V(r) = \int W(r, k, u)k\mathrm{d}k\mathrm{d}u$:

$$\frac{\partial V_i}{\partial r} = \frac{\delta k\,\delta u}{\partial r}\sum_{j,l} k_j(W_{i+1/2,j,l} - W_{i-1/2,j,l}).$$

7. Boundary Conditions

It is necessary to have two fictitious points outside the physical boundaries. On the left of $r = 0$, we use the relation $W(z, -r, k, u) = W(z, r, k, -u)$. On the right of $r = r_{\max}$ we use outgoing boundary conditions. On the left of $k = 0$, we use $W(z, r, -k, u) = W(z, r, k, -u)$. On the right of $k = k_{\max}$ we use outgoing boundary conditions. At $u = \pm 1$, we use the reflecting boundary condition $\partial W / \partial u = 0$.

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REFERENCES

1. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Cambridge, UK, 1995).
2. A. T. Friberg and R. J. Sudol, "Propagation parameters of Gaussian Schell-model beams," Opt. Commun. **41**, 383–387 (1982).
3. F. Gori, "Mode propagation of the field generated by Collett–Wolf Schell-model sources," Opt. Commun. **46**, 149–154 (1983).
4. P. De Santis, F. Gori, G. Guattari, and C. Palma, "An example of a Collett–Wolf source," Opt. Commun. **29**, 256–260 (1979).
5. R. Simon and N. Mukunda, "Twisted Gaussian Schell-model beams," J. Opt. Soc. Am. A **10**, 95–109 (1993).
6. F. Gori, G. Guattari, and C. Padovani, "Modal expansion for J_0-correlated Schell-model sources," Opt. Commun. **64**, 311–316 (1987).
7. F. Gori, M. Santarsiero, R. Borghi, and S. Vicalvi, "Partially coherent sources with helicoidal modes," J. Mod. Opt. **35**, 539–554 (1998).
8. J. Garnier, C. Gouédard, and L. Videau, "Propagation of a partially coherent beam under the interaction of small and large scales," Opt. Commun. **176**, 281–297 (2000).
9. W. Wang and E. Wolf, "Propagation of Gaussian Schell-model beams in dispersive and absorbing media," J. Mod. Opt. **39**, 2007–2021 (1992).
10. B. Gross and J. T. Manassah, "Modification of a quasi-monochromatic beam spatial coherence function through propagation in a two-dimensional Kerr medium," Opt. Lett. **17**, 166–168 (1992).
11. J. Garnier, L. Videau, C. Gouédard, and A. Migus, "Propagation and amplification of incoherent pulses in dispersive and nonlinear media," J. Opt. Soc. Am. B **15**, 2773–2781 (1998).
12. V. P. Nayyar, "Propagation of partially coherent Gaussian Schell-model sources in nonlinear media," J. Opt. Soc. Am. B **14**, 2248–2253 (1997).
13. W. T. Coffey, "On the direct calculation of the Kerr effect of an assembly of dipolar molecules," Chem. Phys. **143**, 171–183 (1990).
14. A. C. Newell and J. W. Moloney, *Nonlinear Optics* (Addison-Wesley, Redwood City, Calif., 1992).
15. J. W. Goodman, "Statistical properties of laser speckle patterns," in *Laser Speckle and Related Phenomena*, J. C. Dainty, ed., Vol. 9 of Topics in Applied physics (Springer-Verlag, Berlin, 1984), pp. 9–75.
16. G. A. Pasmanik, "Self-interaction of incoherent light beams," Sov. Phys. JETP **39**, 234–238 (1974).
17. V. A. Aleshkevich, S. S. Lebedev, and A. N. Matveev, "Self-interaction of a noncoherent light beam," Sov. J. Quantum Electron. **11**, 647–649 (1981).
18. D. Middleton, *Introduction to Statistical Communication Theory* (McGraw-Hill, New York, 1960).
19. R. Adler, *The Geometry of Random Fields* (Wiley, New York, 1981).
20. M. J. Bastiaans, "Application of the Wigner distribution function to partially coherent light," J. Opt. Soc. Am. A **3**, 1227–1238 (1986).
21. R. J. Leveque, *Numerical Methods for Conservation Laws* (Birkhauser, Basel, 1992).