



## Shock structure due to stochastic forcing and the time reversal of nonlinear waves

Jean-Pierre Fouque<sup>a</sup>, Josselin Garnier<sup>b,\*</sup>, André Nachbin<sup>c</sup>

<sup>a</sup> Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA

<sup>b</sup> Laboratoire de Statistique et Probabilités, Université Paul Sabatier, 118 Route de Narbonne, 31062 Toulouse Cedex 4, France

<sup>c</sup> Instituto de Matemática Pura e Aplicada, Est. D Castorina 110, Jardim Botânico, Rio de Janeiro, RJ 22460-320, Brazil

Received 19 August 2003; received in revised form 27 February 2004; accepted 7 May 2004

Communicated by C.K.R.T. Jones

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### Abstract

This paper is concerned with the study of the deformation of a nonlinear pulse traveling in a random medium. We consider shallow water waves with a spatially random depth. We demonstrate that in the presence of properly scaled stochastic forcing the solution to the nonlinear conservation law is regularized leading to a viscous shock profile. This enables us to perform time-reversal experiments beyond the critical time for shock formation.

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PACS: 47.35.+i; 42.25.Dd; 43.25.Ts; 43.60.Cg

Keywords: Nonlinear waves; Inhomogeneous media; Viscous shock; Time reversal

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### 1. Introduction

In this paper, we address the propagation of nonlinear waves in a disordered one-dimensional medium. We demonstrate that in the presence of properly scaled stochastic forcing the solution to the nonlinear conservation law is regularized leading to a viscous shock profile. More precisely, we consider a two-by-two hyperbolic system and we show that to leading order the transmitted wave is governed by a viscous Burgers equation. As a consequence of this regularization effect, time-reversal can be performed beyond the shock critical time.

The propagation of a linear pulse through a random medium has been extensively studied (see for instance the review [2]). In particular the O'Doherty Anstey theory predicts that if the pulse is observed in a Lagrangian frame

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\* Corresponding author. Tel.: +33 56155 6220; fax: +33 56155 6089.

E-mail addresses: [fouque@math.ncsu.edu](mailto:fouque@math.ncsu.edu) (J.-P. Fouque); [garnier@cict.fr](mailto:garnier@cict.fr) (J. Garnier); [nachbin@impa.br](mailto:nachbin@impa.br) (A. Nachbin).

that moves with a random velocity, then the pulse appears to retain its shape up to a slow spreading and attenuation [22]. A rather convincing heuristic explanation of this phenomenon is given in [5]. The mathematical treatment of this issue is addressed in [5,4,7,17,18,3]. We have extended this theory to nonlinear waves by decomposing the solution of the perturbed system using the Riemann invariants of the unperturbed system. Using limit theorems and stochastic calculus we show that the right-going Riemann invariant satisfies a viscous Burgers equation. The apparent viscosity reads as a pseudo-differential operator imposed by the power spectral density of the random fluctuations of the medium.

Time-reversal refocusing for waves propagating in inhomogeneous media have been recently observed and studied experimentally in various contexts, e.g., ultrasound, underwater acoustics, see for instance the review [11]. Important potential applications have been proposed in various fields, for instance imaging [12,23] and communication [10]. A time-reversal mirror is, roughly speaking, a device which is capable of receiving a signal in time, keeping it in memory and sending it back into the medium in the reversed direction of time. The main effect is the refocusing of the scattered signal after time-reversal in a random medium. Surprisingly, the refocused pulse shape only depends on the statistical properties of the random medium, and not on the particular realization of the medium. The full mathematical understanding, meaning both modeling of the physical problem and derivation of the time-reversal effect, is a complex problem. The study of the one-dimensional linear case is now well understood [8,24,14]. In this paper, we give an analysis of time-reversal of weakly nonlinear waves propagating in a one-dimensional disordered medium. The main result is that the regularization effect mentioned above enables us to perform time-reversal of the transmitted wave beyond the shock distance of the homogeneous problem.

This paper is organized as follows. In Section 2, we introduce the nonlinear shallow water wave model with random depth together with the corresponding Riemann invariants. In Section 3, we derive the effective Burgers equation governing the evolution of the front pulse. We discuss the properties of this effective equation in Section 4. We study the time reversal of this pulse in Section 5. Section 6 is devoted to full numerical simulations of the system.

## 2. Shallow water waves with random depth

The shallow water equations are given by [9]

$$\frac{\partial \eta}{\partial t} + \frac{\partial(1 + \varepsilon h + \alpha \eta)u}{\partial x} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial \eta}{\partial x} + \alpha u \frac{\partial u}{\partial x} = 0, \tag{2}$$

where  $\eta$  is the free surface elevation and  $u$  the horizontal velocity component. The fluid body is given by  $H(x, t) = 1 + \varepsilon h(x) + \alpha \eta(x, t)$ . The parameter  $\alpha$  is the ratio of the typical wave amplitude over the mean depth. It governs the strength of the nonlinearity. It is assumed to be small. The parameter  $\varepsilon$  is the order of magnitude of the fluctuations of the depth. It is assumed to be small also. The process  $h$  models the normalized fluctuations of the depth. It is assumed to be a zero-mean, stationary, and ergodic process with  $\mathcal{C}^2$ -paths. The autocorrelation function, and more generally the  $\phi$ -mixing function, should decay fast enough so that it belongs to  $L^1(\mathbb{R}^+)$ . The maximum of  $h$  and its first derivative  $h'$  over an interval of length of order  $\varepsilon^{-2}$

$$M_L^\varepsilon := \sup_{x \in [0, L/\varepsilon^2]} (|h(x)| + |h'(x)|)$$

should be much smaller than  $\varepsilon^{-1}$ . More precisely, we assume that, for any  $L > 0$ ,  $\delta > 0$ ,

$$\mathbb{P}(\varepsilon M_L^\varepsilon \leq \delta) \xrightarrow{\varepsilon \rightarrow 0} 1. \quad (3)$$

This condition is readily fulfilled in realistic configurations, as shown by the following result [1].

**Lemma 2.1.** *If  $h$  is a stationary Gaussian process with a smooth autocorrelation function (say of class  $\mathcal{C}^6$ ), then the supremum  $M_L^\varepsilon$  is of order  $|\ln \varepsilon|^{1/2}$  as  $\varepsilon \rightarrow 0$  (in probability and in expectation).*

For instance, we may think that  $h$  is a Gaussian process with Gaussian autocorrelation function.

We have assumed that  $\varepsilon$  and  $\alpha$  are small. We shall see that the suitable scaling between  $\varepsilon$  and  $\alpha$  to exhibit the interplay between the random and nonlinear effects is

$$\alpha = \varepsilon^2 \alpha_0, \quad (4)$$

where  $\alpha_0$  is the normalized nonlinear parameter which is of order 1. In the physical world it means that the amplitude of the wave elevation is smaller than the amplitude of the fluctuations of the bottom, which is itself smaller than the mean depth.

Let us introduce the “deterministic” local propagation speed

$$c = \sqrt{1 + \alpha\eta}, \quad (5)$$

which does not include the term  $\varepsilon h$ , but it is nevertheless random through the term  $\alpha\eta$ . We can reformulate the above equations in terms of  $c$  and  $u$  to obtain

$$\frac{\partial c}{\partial t} + \frac{\alpha}{2} c \frac{\partial u}{\partial x} + \alpha u \frac{\partial c}{\partial x} + \frac{\alpha \varepsilon}{2c} \frac{\partial hu}{\partial x} = 0, \quad (6)$$

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \frac{2c}{\alpha} \frac{\partial c}{\partial x} = 0. \quad (7)$$

We define the Riemann invariants

$$A(x, t) = \frac{\alpha u - 2c + 2}{\alpha}, \quad (8)$$

$$B(x, t) = \frac{\alpha u + 2c - 2}{\alpha}. \quad (9)$$

If the nonlinearity parameter is vanishing  $\alpha = 0$ , we get back the standard left- and right-going modes ( $A$  and  $B$ , respectively) of the hyperbolic linear system. In presence of nonlinearity and randomness the Riemann invariants satisfy

$$\frac{\partial A}{\partial t} + (\alpha u - c) \frac{\partial A}{\partial x} = \varepsilon \frac{1}{c} \frac{\partial hu}{\partial x}, \quad (10)$$

$$\frac{\partial B}{\partial t} + (\alpha u + c) \frac{\partial B}{\partial x} = -\varepsilon \frac{1}{c} \frac{\partial hu}{\partial x}. \quad (11)$$

Note that in absence of random perturbations  $\varepsilon = 0$ , the two Riemann invariants are constant along different characteristics. We can rewrite these equations in terms of  $A$  and  $B$  only:

$$\frac{\partial A}{\partial t} + \left(-1 + \alpha \frac{3A + B}{4}\right) \frac{\partial A}{\partial x} = \frac{\varepsilon}{2} \frac{\partial h(A + B)}{\partial x} \frac{1}{1 + \alpha(B - A)/4}, \quad (12)$$

$$\frac{\partial B}{\partial t} + \left(1 + \alpha \frac{A + 3B}{4}\right) \frac{\partial B}{\partial x} = -\frac{\varepsilon}{2} \frac{\partial h(A + B)}{\partial x} \frac{1}{1 + \alpha(B - A)/4}. \quad (13)$$

**Remark.** We could have chosen to consider the true “random” local propagation speed

$$\check{c} = \sqrt{1 + \alpha\eta + \varepsilon h}.$$

The Riemann invariants are still defined by

$$\check{A}(x, t) = \frac{\alpha u - 2\check{c} + 2}{\alpha}, \quad \check{B}(x, t) = \frac{\alpha u + 2\check{c} - 2}{\alpha}.$$

They satisfy the following system of equations

$$\frac{\partial \check{A}}{\partial t} + \left(-1 + \alpha \frac{3\check{A} + \check{B}}{4}\right) \frac{\partial \check{A}}{\partial x} = \frac{\varepsilon}{\alpha} \frac{\partial h}{\partial x}, \quad \frac{\partial \check{B}}{\partial t} + \left(1 + \alpha \frac{\check{A} + 3\check{B}}{4}\right) \frac{\partial \check{B}}{\partial x} = -\frac{\varepsilon}{\alpha} \frac{\partial h}{\partial x}.$$

This system seems simpler as the random noise is additive, and not multiplicative. However, we must remember that  $\alpha = \varepsilon^2 \alpha_0$ , so the perturbation is already singular in the microscopic scale ( $t$  and  $x$  of order 1), which leads to enhanced complications in the macroscopic scales ( $t$  and  $x$  of order  $\varepsilon^{-2}$ ). That is why we have adopted the formulation in terms of the “deterministic” local propagation speed  $c$ .

Taking into account the fact that  $\alpha = \alpha_0 \varepsilon^2$ , we can rewrite the evolution equations for  $A$  and  $B$  in the following way, up to terms of order  $\varepsilon^3$ ,

$$\frac{\partial}{\partial x} \begin{pmatrix} A \\ B \end{pmatrix} = Q(x) \frac{\partial}{\partial t} \begin{pmatrix} A \\ B \end{pmatrix} - \varepsilon \frac{h'}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + \varepsilon^2 \frac{\alpha_0}{4} \begin{pmatrix} 3A + B & 0 \\ 0 & A + 3B \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} A \\ B \end{pmatrix} + O(\varepsilon^3), \tag{14}$$

where  $h'$  stands for the spatial derivative of  $h$  and

$$Q(x) = \frac{1}{1 + \varepsilon h} \begin{pmatrix} 1 + \varepsilon \frac{h}{2} & \varepsilon \frac{h}{2} \\ -\varepsilon \frac{h}{2} & -1 - \varepsilon \frac{h}{2} \end{pmatrix} \tag{15}$$

The system is completed by the initial condition corresponding to a right-going wave incoming from the homogeneous half-space  $x < 0$

$$A(x, t) = 0, \quad B(x, t) = f(t - x), \quad t < 0,$$

where the function  $f$  is compactly supported in  $(0; \infty)$ .

### 3. Derivation of the effective equation for the front pulse

In this section, we perform a series of transformations to rewrite the evolution equations of the modes by centering along the characteristic of the right-going mode. We will then obtain a lower-triangular system that can be integrated more easily. In a second step we shall apply an averaging theorem to this system to establish an effective nonlinear equation for the front pulse.

The eigenvalues of the matrix  $Q(x)$  are  $\pm \gamma^\varepsilon(x)$  with

$$\gamma^\varepsilon(x) = \frac{1}{\sqrt{1 + \varepsilon h}}. \tag{16}$$

We introduce the unitary matrix  $U$

$$U = \sqrt{\frac{2 + \varepsilon h + 2\sqrt{1 + \varepsilon h}}{4 + 2\varepsilon h}} \begin{pmatrix} 1 & -\frac{\varepsilon h}{2 + \varepsilon h + 2\sqrt{1 + \varepsilon h}} \\ -\frac{\varepsilon h}{2 + \varepsilon h + 2\sqrt{1 + \varepsilon h}} & 1 \end{pmatrix}, \quad (17)$$

which is such that

$$UQU^{-1} = \gamma^\varepsilon(x) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We then introduce the modified modes

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = U \begin{pmatrix} A \\ B \end{pmatrix}. \quad (18)$$

They satisfy

$$\begin{aligned} \frac{\partial}{\partial x} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} &= \gamma^\varepsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} - \varepsilon \frac{h'}{4} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} \\ &+ \varepsilon^2 \frac{\alpha_0}{4} \begin{pmatrix} 3A_1 + B_1 & 0 \\ 0 & A_1 + 3B_1 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} + O(\varepsilon^3). \end{aligned} \quad (19)$$

We consider the new variable

$$z(x) = \int_0^x \gamma^\varepsilon(x') dx' \quad (20)$$

and we again modify slightly the modes to simplify the equation:

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix}(z, t) = \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}(x(z), t) \exp\left(\varepsilon \frac{h(x(z))}{2}\right). \quad (21)$$

Note that  $h'(x(z))/\gamma(x(z)) = (d/dz)h(x(z))$ . We still denote this quantity by  $h'$ , so  $A_2$  and  $B_2$  satisfy

$$\begin{aligned} \frac{\partial}{\partial z} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} - \varepsilon \frac{h'}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} \\ &+ \varepsilon^2 \frac{\alpha_0}{4} \begin{pmatrix} 3A_2 + B_2 & 0 \\ 0 & A_2 + 3B_2 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} + O(\varepsilon^3). \end{aligned} \quad (22)$$

We consider the reference frame that moves with the right-going mode  $B_2$

$$\tau = t - z, \quad (23)$$

so that the equation for  $(A_2, B_2)$  now reads

$$\frac{\partial}{\partial z} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \frac{\partial}{\partial \tau} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} - \varepsilon \frac{h'}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} + \varepsilon^2 \frac{\alpha_0}{4} \begin{pmatrix} 3A_2 + B_2 & 0 \\ 0 & A_2 + 3B_2 \end{pmatrix} \frac{\partial}{\partial \tau} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} + O(\varepsilon^3). \quad (24)$$

The equation for  $B_2$  can be integrated as

$$B_2(z, \tau) = \int_0^z S_B(y, \tau) dy + f(\tau), \tag{25}$$

$$S_B(y, \tau) = -\varepsilon \frac{h'(y)}{4} A_2(y, \tau) + \varepsilon^2 \frac{\alpha_0}{4} (A_2 + 3B_2) \frac{\partial B_2}{\partial \tau}(y, \tau) + O(\varepsilon^3). \tag{26}$$

Note that we consider propagation distance  $z$  of order  $\varepsilon^{-2}$ , but as we are interested in the front pulse we only consider local times  $\tau$  of order 1, lying in some interval  $[0, T]$  with fixed  $T$ . From now on we need to take precaution to get precise estimates because of the shock forming nature of the equations. Let  $K > 0$ . We introduce the stopping distance

$$L_K = \inf \left\{ l \geq 0, \sup_{\tau \in [0, T]} \left| B_2 \left( \frac{l}{\varepsilon^2}, \tau \right) \right| + \left| \frac{\partial B_2}{\partial \tau} \left( \frac{l}{\varepsilon^2}, \tau \right) \right| \geq K \right\}. \tag{27}$$

As long as  $z \leq L_K/\varepsilon^2$ , the solutions of the Eq. (24) are well-defined. More precisely, the equation for  $A_2$  can be integrated as

$$A_2(z, \tau) = -\frac{1}{2} \int_{-\infty}^{\tau} S_A \left( z + \frac{\tau - s}{2}, s \right) ds, \tag{28}$$

$$S_A(z, s) = -\varepsilon \frac{h'(z)}{4} B_2(z, s) + \varepsilon^2 \frac{\alpha_0}{4} (3A_2 + B_2) \frac{\partial A_2}{\partial \tau}(z, s) + O(\varepsilon^3). \tag{29}$$

The initial conditions impose that  $A_2$  and  $B_2$  are zero for  $\tau < 0$  and  $z = 0$ . The transport Eq. (24) then show that  $A_2$  and  $B_2$  are zero for  $\tau < 0$  whatever  $z \geq 0$ . Thus, the integral with respect to  $s$  in Eq. (28) actually goes from 0 to  $\tau$ . Furthermore, Eq. (29) shows that  $S_A$  is of order  $\varepsilon$ . This allows us to claim that

$$\sup_{z \in [0, L_K/\varepsilon^2], \tau \in [0, T]} |A_2(z, \tau)| \leq K' \varepsilon \tag{30}$$

and

$$\sup_{z \in [0, L_K/\varepsilon^2], \tau \in [0, T]} \left| \frac{\partial A_2}{\partial \tau}(z, \tau) \right| \leq K' \varepsilon. \tag{31}$$

Furthermore, we can substitute the integral representation of  $A_2$  into the one of  $B_2$

$$\begin{aligned} B_2(z, \tau) &= f(\tau) - \frac{\varepsilon^2}{32} \int_0^z h'(y) \int_0^{\tau} h' \left( y + \frac{\tau - s}{2} \right) B_2 \left( y + \frac{\tau - s}{2}, s \right) ds dy \\ &\quad + \varepsilon^2 \frac{3\alpha_0}{4} \int_0^z B_2 \frac{\partial B_2}{\partial \tau}(y, \tau) dy + O(\varepsilon^3(1 + z)). \end{aligned} \tag{32}$$

Note that we have eliminated the terms  $\varepsilon^2 A_2 \partial_{\tau} B_2$ ,  $\varepsilon^2 A_2 \partial_{\tau} A_2$ ,  $\varepsilon^2 B_2 \partial_{\tau} A_2$ , as they are of order  $\varepsilon^3$  and are negligible for propagation distance of order  $\varepsilon^{-2}$ .

We introduce

$$B_2^{\varepsilon}(z, \tau) = B_2 \left( \frac{z}{\varepsilon^2}, \tau \right),$$

which satisfies

$$B_2^\varepsilon(z, \tau) = f(\tau) - \frac{1}{32} \int_0^z h' \left( \frac{y}{\varepsilon^2} \right) \int_0^\tau h' \left( \frac{y}{\varepsilon^2} + \frac{\tau - s}{2} \right) B_2^\varepsilon \left( y + \varepsilon^2 \frac{\tau - s}{2}, s \right) ds dy + \frac{3\alpha_0}{4} \int_0^z B_2^\varepsilon \frac{\partial B_2^\varepsilon}{\partial \tau}(y, \tau) dy + O(\varepsilon). \tag{33}$$

By applying the Khasminskii’s averaging theorem [16], we get that  $B_2^\varepsilon$  converges to  $\tilde{B}_2$  solution of

$$\tilde{B}_2(z, \tau) = f(\tau) - \frac{1}{16} \int_0^z \Lambda \tilde{B}_2(y, \tau) dy + \frac{3\alpha_0}{4} \int_0^z \tilde{B}_2 \frac{\partial \tilde{B}_2}{\partial \tau}(y, \tau) dy, \tag{34}$$

where the operator  $\Lambda$  is

$$\Lambda F(\tau) = \frac{1}{2} \int_0^\tau \phi_1 \left( \frac{s}{2} \right) F(\tau - s) ds = \left[ \frac{1}{2} \phi_1 \left( \frac{\cdot}{2} \right) \mathbf{1}_{[0, \infty)}(\cdot) \right] * F(\tau), \tag{35}$$

$$\phi_1(y) = \mathbb{E}[h'(z)h'(z + y)]. \tag{36}$$

The convergence holds true in the space of the continuous functions  $\mathcal{C}([0, \tilde{L}_K] \times [0, T], \mathbb{R})$  where  $T$  and  $K$  are arbitrary, and  $\tilde{L}_K$  is the stopping distance defined by (27) for the deterministic function  $\tilde{B}_2$ . In the Fourier domain

$$\int_{-\infty}^\infty \Lambda f(\tau) e^{i\omega\tau} d\tau = b_1(2\omega) \int_{-\infty}^\infty f(\tau) e^{i\omega\tau} d\tau, \tag{37}$$

$$b_1(\omega) = \int_0^\infty \phi_1(\tau) e^{i\omega\tau} d\tau, \tag{38}$$

and it is straightforward to check by integrating by parts that

$$b_1(\omega) = -i\omega\phi_0(0) + \omega^2 b_0(\omega), \tag{39}$$

$$b_0(\omega) = \int_0^\infty \phi_0(y) e^{i\omega y} dy, \tag{40}$$

$$\phi_0(y) = \mathbb{E}[h(z)h(z + y)]. \tag{41}$$

Accordingly,

$$\frac{\partial \tilde{B}_2}{\partial z} = -\frac{\phi_0(0)}{8} \frac{\partial \tilde{B}_2}{\partial \tau} + \mathcal{L} \tilde{B}_2 + \frac{3\alpha_0}{4} \tilde{B}_2 \frac{\partial \tilde{B}_2}{\partial \tau}, \tag{42}$$

where  $\mathcal{L}$  is a pseudo-differential operator.  $\mathcal{L}$  behaves like a diffusion for small frequencies, but it decays to zero for high-frequencies signals. Its symbol is

$$\mathcal{L} = -\frac{b_0(2\omega)\omega^2}{4}, \tag{43}$$

which means that in the Fourier domain,

$$\int_{-\infty}^\infty \mathcal{L} f(\tau) e^{i\omega\tau} d\tau = -\frac{b_0(2\omega)\omega^2}{4} \int_{-\infty}^\infty f(\tau) e^{i\omega\tau} d\tau.$$

In terms of the true mode  $B$ , we have to take care of the change of variable  $x \mapsto z(x)$ . In the macroscopic scales

$$z \left( \frac{x}{\varepsilon^2} \right) = \frac{x}{\varepsilon^2} - \frac{\varepsilon}{2} \int_0^{x/\varepsilon^2} h(x') dx' + \frac{\varepsilon^2}{8} \int_0^{x/\varepsilon^2} h(x')^2 dx' + O(\varepsilon), \tag{44}$$

so that it converges in distribution as

$$z\left(\frac{x}{\varepsilon^2}\right) - \frac{x}{\varepsilon^2} \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{\sqrt{2}}\sqrt{b_0(0)}W_x + \frac{1}{8}\phi_0(0)x, \tag{45}$$

where  $W_x$  is a standard Brownian motion. We can then state the following proposition.

**Proposition 3.1.** *Let  $\tilde{B}_0$  be the solution of*

$$\frac{\partial \tilde{B}_0}{\partial x} = \mathcal{L}\tilde{B}_0 + \frac{3\alpha_0}{4}\tilde{B}_0 \frac{\partial \tilde{B}_0}{\partial \tau} \tag{46}$$

starting from  $\tilde{B}_0(0, \tau) = f(\tau)$ . We denote by  $L_{shock}$  the shock distance of  $\tilde{B}_0$ . For any  $L < L_{shock}$ , the front pulse  $B^\varepsilon(x, \tau) := B(x/\varepsilon^2, \tau + x/\varepsilon^2)$ ,  $x \in [0, L]$ , converges in distribution in the space of the continuous functions to  $\tilde{B}$  given by

$$\tilde{B}(x, \tau) = \tilde{B}_0\left(x, \tau - \frac{\sqrt{b_0(0)}}{\sqrt{2}}W_x - \frac{\phi_0(0)}{4}x\right). \tag{47}$$

#### 4. Analysis of the equation for the front pulse

In this section, we analyse the main properties of the effective equation for the front pulse. The effective equation for the front pulse depends both on randomness (through the function  $b_0$ ) and on nonlinearity (through the parameter  $\alpha_0$ ). The important function that imposes the dynamics is the Fourier transform of the positive lag part of the autocorrelation function of the random fluctuations of the bottom

$$b_0(\omega) = \int_0^\infty \mathbb{E}[h(0)h(x)] \exp(i\omega x) dx.$$

Note that  $b_0(0)$  is proportional to the power spectral density of the random process  $h$  and is therefore a nonnegative real number. We have proved that  $B(x/\varepsilon^2, x/\varepsilon^2 + t)$  converges to  $\tilde{B}$  given by (46) and (47). This convergence holds true till the shock formation distance of the deterministic limit profile  $\tilde{B}_0$ . The effective Eq. (46) can thus be used to determine the shock distance of the original function in the case  $\varepsilon \ll 1$ , but it cannot be used to go beyond this distance.

In Eqs. (46) and (47), the Brownian motion  $W_x$  corresponds to a random time shift and  $\mathcal{L}$  is a pseudo-differential operator that models the pulse spreading imposed by the random fluctuations of the bottom.  $\mathcal{L}$  behaves like a diffusion for small frequencies, but it decays to zero for high-frequencies. Indeed the proof of Proposition 3.1 requires uniform bounds for  $h'$  (see the estimate (3)). A condition of the type  $\mathbb{E}[h'(x)^2] < \infty$  is equivalent to  $\int \omega^2 Re(b_0(\omega)) d\omega < \infty$ . Accordingly the effective Eq. (46) shows that random fluctuations regularize the shock formation, in the sense that the shock distance is larger in a random medium than in a uniform one. However, the shock may eventually show up. We shall illustrate this regularization effect in Section 6.

The pseudo-spectral operator  $\mathcal{L}$  can be divided into two parts

$$\mathcal{L} = \mathcal{L}_r + \mathcal{L}_i, \tag{48}$$

$$\mathcal{L}_r = -\frac{1}{4}(b_r(2\omega)\omega^2), \tag{49}$$

$$\mathcal{L}_i = -\frac{1}{4}(ib_i(2\omega)\omega^2), \tag{50}$$



where  $b_r$  and  $b_i$  are, respectively, the real and imaginary part of  $b_0$

$$b_r(\omega) = \int_0^\infty \mathbb{E}[h(0)h(x)] \cos(\omega x) dx, \quad (51)$$

$$b_i(\omega) = \int_0^\infty \mathbb{E}[h(0)h(x)] \sin(\omega x) dx. \quad (52)$$

In particular  $b_r$  is proportional to the power spectral density of the random process  $h$ .  $\mathcal{L}_r$  can be described as an effective diffusion operator, while  $\mathcal{L}_i$  is an effective dispersion operator, since it preserves the energy.

Let us address the case where the power spectral density of the process  $h$  can be considered as constant over the spectral range of  $f$ :  $b_0(\omega) \equiv b_0$ . This arises if the typical wavelength of the pulse is larger than the correlation length of the medium. In such a case the early steps of the effective evolution equation is that of the viscous Burgers equation

$$\frac{\partial \tilde{B}_0}{\partial x} = b_0 \frac{\partial^2 \tilde{B}_0}{\partial r^2} + \frac{3\alpha_0}{4} \tilde{B}_0 \frac{\partial \tilde{B}_0}{\partial r}. \quad (53)$$

However, new frequencies are generated by the nonlinear term, that may fall in the tail of the function  $b_0$ . Then the last equation may eventually fail describing exactly the dynamics of the front pulse, and one must consider the true equation with the pseudo-differential operator when a shock starts forming. As pointed out above, we cannot consider the true white noise case, because a white noise does not fulfill the requirement (3) and Proposition 3.1 cannot be proved in such a configuration.

The viscous Burgers Eq. (53) supports self-similar waves or even traveling waves as shown in [26, Chapter 4]. This means that the generalized O'Doherty–Anstey theory in a nonlinear scenario is not necessarily associated with attenuation. In a linear framework the effective viscosity generated by the random fluctuations always give rise to attenuation, but a nonlinearity can compensate for it and even produces a traveling wave that will be stable over a propagation distance that can be substantially larger than the homogeneous shock formation distance. A simple example is the dam-breaking problem, where an initial step propagates into the random medium. The corresponding traveling wave is given in [26, Section 4.3].

## 5. Time reversal of the front pulse

### 5.1. Time reversal set-up

The time reversal procedure consists in recording the transmitted signal at  $z = L/\varepsilon^2$  over the time interval  $[L/\varepsilon^2 + t_0, L/\varepsilon^2 + t_1]$ . A piece of the recorded signal is cut using a cut-off function  $s \mapsto G_{t_0, t_1}(s - L/\varepsilon^2)$  where the support of  $G_{t_0, t_1}$  is included in  $[t_0, t_1]$ :

$$B_{\text{cut}}^\varepsilon(t) = B\left(x = \frac{L}{\varepsilon^2}, \frac{L}{\varepsilon^2} + t\right) G_{t_0, t_1}(t).$$

One then time reverses that piece of signal and send it back into the same medium by changing the sign of the velocity. From the expression of the Riemann invariants (8) and (9), the left going mode sent back is given by

$$C\left(x = \frac{L}{\varepsilon^2}, t\right) = -B\left(x = \frac{L}{\varepsilon^2}, \frac{L}{\varepsilon^2} + t_1 - t\right) G_{t_0, t_1}(t_1 - t).$$



Fig. 1. Forward scattering problem.

In the time reversal experiment, the left-going mode is denoted by  $C$  and the right-going mode by  $D$ :

$$C(x, t) = \frac{\alpha u - 2c + 2}{\alpha}, \quad D(x, t) = \frac{\alpha u + 2c - 2}{\alpha}.$$

**Remark.** To record the transmitted signal, we do not need to record both the elevation  $\eta$  and the velocity  $u$ . If we record only  $\eta$ , then we know the expression of  $u$  because we know that it is a right-going mode:

$$u = \frac{2\sqrt{1 + \alpha\eta} - 2}{\alpha}.$$

Furthermore, to re-emit the signal  $C(x = L/\varepsilon^2, t)$ , we do not need to generate both the elevation and the velocity. Indeed, if we generate for instance twice the elevation with a zero velocity, then this input pulse will split into a right-going mode, that will go to the right without coming back, and a left-going mode that is exactly the desired  $C(x = L/\varepsilon^2, t)$ .

The modes  $C$  and  $D$  satisfy

$$\frac{\partial}{\partial x} \begin{pmatrix} C \\ D \end{pmatrix} = Q(x) \frac{\partial}{\partial t} \begin{pmatrix} C \\ D \end{pmatrix} - \varepsilon \frac{h'}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} + \varepsilon^2 \frac{\alpha_0}{4} \begin{pmatrix} 3C + D & 0 \\ 0 & C + 3D \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} C \\ D \end{pmatrix} + O(\varepsilon^3), \quad (54)$$

with the initial condition for  $x > L/\varepsilon^2$ :

$$C(x, t) = g^\varepsilon \left( t - x + \frac{L}{\varepsilon^2} \right), \quad D(x, t) = 0, \quad t < 0.$$

Note that  $g^\varepsilon$  is compactly supported in  $(0, \infty)$ :

$$g^\varepsilon(t) = -B \left( x = \frac{L}{\varepsilon^2}, \frac{L}{\varepsilon^2} + t_1 - t \right) G_{t_0, t_1}(t_1 - t). \quad (55)$$

The goal of this section is to determine the shape of the transmitted wave at  $x = 0$  around the expected arrival time  $L/\varepsilon^2 + t_1$  (Figs. 1 and 2):

$$C_{tr}^\varepsilon(t) = C \left( x = 0, \frac{L}{\varepsilon^2} + t_1 + t \right). \quad (56)$$

### 5.2. Derivation of the effective equation for the refocused pulse

As in Section 3 we rewrite the evolution of the time-reverted wave by centering along the characteristic of the left-going mode. Here we obtain an upper triangular system. The integration and averaging follow the same lines as in Section 3.



Fig. 2. Time reversal set-up.

We introduce the modified modes

$$\begin{pmatrix} C_2 \\ D_2 \end{pmatrix} (z, t) = U \begin{pmatrix} C \\ D \end{pmatrix} (x(z), t) \exp\left(\varepsilon \frac{h(x(z))}{2}\right), \tag{57}$$

which satisfy

$$\begin{aligned} \frac{\partial}{\partial z} \begin{pmatrix} C_2 \\ D_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} C_2 \\ D_2 \end{pmatrix} - \varepsilon \frac{h'}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C_2 \\ D_2 \end{pmatrix} \\ &+ \varepsilon^2 \frac{\alpha_0}{4} \begin{pmatrix} 3C_2 + D_2 & 0 \\ 0 & C_2 + 3D_2 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} C_2 \\ D_2 \end{pmatrix} + O(\varepsilon^3). \end{aligned} \tag{58}$$

We consider the reference frame that moves with the left-going mode  $C_2$

$$\tau = t + z - z \left( \frac{L}{\varepsilon^2} \right), \tag{59}$$

so that the equation now reads

$$\begin{aligned} \frac{\partial}{\partial z} \begin{pmatrix} C_2 \\ D_2 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \frac{\partial}{\partial \tau} \begin{pmatrix} C_2 \\ D_2 \end{pmatrix} - \varepsilon \frac{h'}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C_2 \\ D_2 \end{pmatrix} \\ &+ \varepsilon^2 \frac{\alpha_0}{4} \begin{pmatrix} 3C_2 + D_2 & 0 \\ 0 & C_2 + 3D_2 \end{pmatrix} \frac{\partial}{\partial \tau} \begin{pmatrix} C_2 \\ D_2 \end{pmatrix} + O(\varepsilon^3). \end{aligned} \tag{60}$$

Integrating the equation for  $D_2$  along its characteristic and substituting into the integral representation of  $C_2$  establishes that, for  $z \in [0, z(L/\varepsilon^2)]$ ,

$$\begin{aligned} C_2(z, \tau) &= g^\varepsilon(\tau) - \frac{\varepsilon^2}{32} \int_z^{z(L/\varepsilon^2)} h'(y) \int_0^\tau h' \left( y - \frac{\tau-s}{2} \right) C_2 \left( y - \frac{\tau-s}{2}, s \right) ds dy \\ &- \varepsilon^2 \frac{3\alpha_0}{4} \int_z^{z(L/\varepsilon^2)} C_2 \frac{\partial C_2}{\partial \tau} (y, \tau) dy + O(\varepsilon). \end{aligned} \tag{61}$$

Introducing

$$C_3(z, \tau) = C_2 \left( z \left( \frac{L}{\varepsilon^2} \right) - z, \tau \right),$$

we get the equation

$$C_3(z, \tau) = g^\varepsilon(\tau) - \frac{\varepsilon^2}{32} \int_0^z \int_0^\tau \left[ h' \left( z \left( \frac{L}{\varepsilon^2} \right) - y \right) h' \left( z \left( \frac{L}{\varepsilon^2} \right) - y - \frac{\tau - s}{2} \right) C_3 \left( y - \frac{\tau - s}{2}, s \right) \right] ds dy - \varepsilon^2 \frac{3\alpha_0}{4} \int_0^z C_3 \frac{\partial C_3}{\partial \tau}(y, \tau) dy + O(\varepsilon).$$

The refocused pulse in terms of  $C_3$  reads as

$$C_{tr}^\varepsilon(\tau) = C_3 \left( z \left( \frac{L}{\varepsilon^2} \right), \tau + t_1 + \frac{L}{\varepsilon^2} - z \left( \frac{L}{\varepsilon^2} \right) \right) + O(\varepsilon).$$

Remember also that

$$g^\varepsilon(\tau) = -B_2 \left( z \left( \frac{L}{\varepsilon^2} \right), \frac{L}{\varepsilon^2} - z \left( \frac{L}{\varepsilon^2} \right) + t_1 - \tau \right) G_{t_0, t_1}(t_1 - \tau) + O(\varepsilon).$$

We then shift and invert  $C_3$ :

$$C_4(z, \tau) = -C_3 \left( z, t_1 + \frac{L}{\varepsilon^2} - z \left( \frac{L}{\varepsilon^2} \right) - \tau \right).$$

$C_4$  satisfies

$$C_4(z, \tau) = g_4^\varepsilon(\tau) - \frac{\varepsilon^2}{32} \int_0^z \int_\tau^\infty \left[ h' \left( z \left( \frac{L}{\varepsilon^2} \right) - y \right) h' \left( z \left( \frac{L}{\varepsilon^2} \right) - y + \frac{\tau - s}{2} \right) C_4 \left( y - \frac{s}{2}, s \right) \right] ds dy - \varepsilon^2 \frac{3\alpha_0}{4} \int_0^z C_4 \frac{\partial C_4}{\partial \tau}(y, \tau) dy + O(\varepsilon), \tag{62}$$

where

$$g_4^\varepsilon(\tau) = B_2 \left( z \left( \frac{L}{\varepsilon^2} \right), \tau \right) G_{t_0, t_1} \left( z \left( \frac{L}{\varepsilon^2} \right) - \frac{L}{\varepsilon^2} + \tau \right) + O(\varepsilon).$$

The integral over  $s$  in Eq. (62) actually goes from  $\tau$  to  $t_1 + L/\varepsilon^2 - z(L/\varepsilon^2)$  because  $C_4$  vanishes for  $\tau > t_1 + L/\varepsilon^2 - z(L/\varepsilon^2)$ . The refocused pulse in terms of  $C_4$  reads as

$$C_{tr}^\varepsilon(\tau) = -C_4 \left( z \left( \frac{L}{\varepsilon^2} \right), -\tau \right) + O(\varepsilon).$$

We can now apply the averaging theorem. We get that  $C_4$  converges as  $\varepsilon \rightarrow 0$  to  $\tilde{C}_4$  solution of

$$\tilde{C}_4(z, \tau) = g(\tau) - \frac{1}{16} \int_0^z \Lambda^* \tilde{C}_4(y, \tau) dy + \frac{3\alpha_0}{4} \int_0^z \tilde{C}_4 \frac{\partial \tilde{C}_4}{\partial \tau}(y, \tau) dy, \tag{63}$$

where the function  $g$  is

$$g(\tau) = \tilde{B}_0 \left( L, \tau - \frac{\phi_0(0)}{8} L \right) G_{t_0, t_1} \left( \tau + \frac{\sqrt{b_0(0)}}{\sqrt{2}} W_L + \frac{\phi_0(0)}{8} L \right) \tag{64}$$

and the operator  $\Lambda^*$  is

$$\Lambda^* f(\tau) = \frac{1}{2} \int_0^\tau \phi_1\left(-\frac{s}{2}\right) f(\tau - s) ds = \left[ \frac{1}{2} \phi_1\left(-\frac{\cdot}{2}\right) \mathbf{1}_{[0, \infty)}(\cdot) \right]^* f(\tau). \tag{65}$$

In the Fourier domain

$$\int_{-\infty}^\infty \Lambda^* f(\tau) e^{i\omega\tau} d\tau = \overline{b_1(2\omega)} \int_{-\infty}^\infty f(\tau) e^{i\omega\tau} d\tau. \tag{66}$$

Accordingly,

$$\frac{\partial \tilde{C}_4}{\partial z} = \frac{\phi_0(0)}{8} \frac{\partial \tilde{C}_4}{\partial \tau} + \mathcal{L}^* \tilde{C}_4 - \frac{3\alpha_0}{4} \tilde{C}_4 \frac{\partial \tilde{C}_4}{\partial \tau}, \tag{67}$$

where  $\mathcal{L}^* = \mathcal{L}_r - \mathcal{L}_i$  and  $\mathcal{L}_r$  and  $\mathcal{L}_i$  are defined by Eqs. (49) and (50). The refocused pulse  $C_{tr}^\varepsilon$  also converges to

$$\tilde{C}_{tr}(\tau) = -\tilde{C}_4(L, -\tau). \tag{68}$$

Performing a final time shift allows us to state the following proposition.

**Proposition 5.1.** *The refocused pulse  $C_{tr}^\varepsilon$  converges as  $\varepsilon \rightarrow 0$  to*

$$\tilde{C}_{tr}(\tau) = -\tilde{C}_0(L, -\tau), \tag{69}$$

where

$$\frac{\partial \tilde{C}_0}{\partial z} = \mathcal{L}_r \tilde{C}_0 - \mathcal{L}_i \tilde{C}_0 - \frac{3\alpha_0}{4} \tilde{C}_0 \frac{\partial \tilde{C}_0}{\partial \tau}, \tag{70}$$

$$\tilde{C}_0(0, \tau) = \tilde{B}_0(L, \tau) G_{t_0, t_1}(\tau + T_d), \tag{71}$$

$$T_d = \frac{\sqrt{b_0(0)}}{\sqrt{2}} W_L + \frac{\phi_0(0)}{4} L, \tag{72}$$

$$\frac{\partial \tilde{B}_0}{\partial z} = \mathcal{L}_r \tilde{B}_0 + \mathcal{L}_i \tilde{B}_0 + \frac{3\alpha_0}{4} \tilde{B}_0 \frac{\partial \tilde{B}_0}{\partial \tau}, \tag{73}$$

$$\tilde{B}_0(0, \tau) = f(\tau), \tag{74}$$

and the pseudo-differential operators  $\mathcal{L}_r$  and  $\mathcal{L}_i$  are defined by Eqs. (49) and (50).

### 5.3. Random linear case

In this subsection we assume that the nonlinear parameter  $\alpha_0 = 0$ . This case is already known in the white noise case [14], so we shall just remember the reader with this result and also extend it to the colored case. Taking the Fourier transform in Eqs. (70–73) yields that the transmitted pulse is

$$\tilde{B}(t) = K_{ODA}^* f(t - T_d), \tag{75}$$

where the convolution kernel  $K_{ODA}$  has Fourier transform

$$\hat{K}_{ODA}(\omega) = e^{-b_r(2\omega)\omega^2 L/4} e^{-ib_i(2\omega)\omega^2 L/4}. \tag{76}$$

The refocused pulse is

$$-\tilde{C}_{tr}(-t) = \frac{1}{2\pi} \int \int e^{-i\omega t} e^{-b_0(2\omega)\omega^2 L/4 - \overline{b_0(2\omega')}\omega'^2 L/4} e^{-i(\omega - \omega')T_d} \times \hat{G}_{t_0, t_1}(\omega - \omega') \hat{f}(\omega') d\omega' d\omega$$

where  $\hat{G}_{t_0,t_1}$ , resp.  $\hat{f}$ , is the Fourier transform of  $G_{t_0,t_1}$ , resp.  $f$ . If the front is completely recorded, then we can substitute 1 for  $G_{t_0,t_1}$  and  $\delta_0$  for  $\hat{G}_{t_0,t_1}$ , so that the refocused pulse is:

$$-\tilde{C}_{tr}(-t) = K_{tr} * f(t) \tag{77}$$

where the convolution kernel has Fourier transform

$$\hat{K}_{tr}(\omega) = e^{-b_r(2\omega)\omega^2 L/2}. \tag{78}$$

The important statement is that the contribution of the random time delay and of the effective dispersion  $\mathcal{L}_i$  have been canceled by the time reversal, but the contribution of the effective diffusion  $\mathcal{L}_r$  has been multiplied by 2.

#### 5.4. Homogeneous nonlinear case

In the homogeneous case, the dynamics of the pulses  $\tilde{B}_0$  and  $\tilde{C}_0$  are governed by the inviscid Burgers equations:

$$\frac{\partial \tilde{C}_0}{\partial z} = -\frac{3\alpha_0}{4} \tilde{C}_0 \frac{\partial \tilde{C}_0}{\partial \tau}, \tag{79}$$

$$\tilde{C}_0(0, \tau) = \tilde{B}_0(L, \tau)G_{t_0,t_1}(\tau), \tag{80}$$

$$\frac{\partial \tilde{B}_0}{\partial z} = \frac{3\alpha_0}{4} \tilde{B}_0 \frac{\partial \tilde{B}_0}{\partial \tau}, \tag{81}$$

$$\tilde{B}_0(0, \tau) = f(\tau). \tag{82}$$

These equations can be solved by the characteristics method [9]. If  $L < L_{shock}$

$$L_{shock} = \frac{4}{3\alpha_0} \min_{\tau, f'(\tau) > 0} \left\{ \frac{1}{f'(\tau)} \right\},$$

then the forward Eq. (81) has a well-defined solution  $\tilde{B}_0$  which satisfies for any  $\tau$ :

$$\tilde{B}_0 \left( L, \tau - \frac{3\alpha_0}{4} f(\tau)L \right) = f(\tau).$$

Similarly, if  $L < L_{shock,2}$ ,

$$L_{shock,2} = \frac{4}{3\alpha_0} \min_{\tau, g'(\tau) > 0} \left\{ -\frac{1}{g'(\tau)} \right\}, \quad g(\tau) = \tilde{B}_0(L, \tau)G_{t_0,t_1}(\tau),$$

then the backward Eq. (79) has a well-defined solution  $\tilde{C}_0$  which satisfies for any  $\tau'$ :

$$\tilde{C}_0 \left( L, \tau' + \frac{3\alpha_0}{4} \tilde{B}_0(L, \tau')G_{t_0,t_1}(\tau')L \right) = \tilde{B}_0(L, \tau')G_{t_0,t_1}(\tau').$$

Choosing  $\tau' = \tau - (3\alpha_0/4)f(\tau)L$  in the last equation:

$$\tilde{C}_0 \left[ L, \tau + \frac{3\alpha_0}{4} f(\tau) \left( G_{t_0,t_1} \left( \tau - \frac{3\alpha_0}{4} f(\tau)L \right) - 1 \right) L \right] = f(\tau)G_{t_0,t_1} \left( \tau - \frac{3\alpha_0}{4} f(\tau)L \right).$$

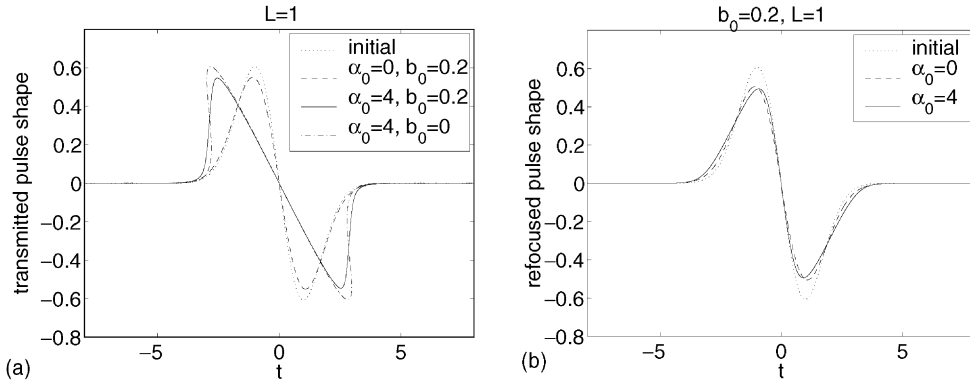


Fig. 3. Transmitted pulse shape  $\tilde{B}_0(L, \cdot)$  (picture a) and refocused pulse shape  $-\tilde{C}_{tr}(\cdot)$  (picture b). We assume that the front pulse is completely recorded. The initial pulse is the derivative of a Gaussian  $f(t) = -t \exp(-t^2/2)$ . We compare the nonlinear configuration  $\alpha_0 = 4$  with the corresponding linear configuration  $\alpha_0 = 0$ . The parameter  $b \equiv 0.2$ .

Accordingly, if  $\tau$  is such that  $G_{t_0, t_1}(\tau - (3\alpha_0/4)f(\tau)L) = 1$ , then  $\tilde{C}_0(L, \tau) = f(\tau)$ .

In the particular case where the front is completely recorded, then  $L_{shock,2} = L_{shock}$  and  $L < L_{shock} \implies \tilde{C}_0(L, \cdot) \equiv f$ . In the homogeneous case, either the propagation distance is smaller than the shock distance, and then time reversal completely rebuilds the initial pulse shape. Or the propagation distance is larger than the shock distance, and then time reversal completely fails as there is no uniqueness of the solution.

### 5.5. Random nonlinear case

To simplify the analysis we shall assume that the power spectral density of the process  $h$  can be considered as constant over the spectral range of  $f$ :  $b_0(\omega) \equiv b_0$ . We also consider that we completely record the transmitted front pulse. In this configuration, applying the Cole–Hopf transformation [9], we get a closed-form expression for the refocused pulse

$$-\tilde{C}_{tr}(-t) = -\frac{2b_0}{3\alpha_0} \left[ \ln \left( \int_{-\infty}^{\infty} \frac{e^{-(s-t)^2/b_0L}}{\int_{-\infty}^{\infty} e^{3\alpha_0/2b_0 \int_0^{s'} f(s'') ds''} e^{-(s-s')^2/b_0L} ds'} ds \right) \right]_t \tag{83}$$

where  $[\cdot]_t$  stands for a differentiation with respect to  $t$ .

Consider Figs. 3–5. We address cases where the transmitted pulse is strongly affected by nonlinearity. In particular the dot-dashed lines in part (a) of Figs. 3–5 plot the transmitted pulse shapes in absence of randomness computed from the characteristics method, which show that a shock would have arisen before reaching the mirror. In such configurations refocusing by time reversal cannot be achieved as uniqueness of the solution is lost. In presence of randomness the effective diffusion prevents from shock formation (solid lines in part (a) of Figs. 3–5). It then appears that the time reversal process succeeds in compensating for the nonlinear effect (solid lines, Figs. 3b and 4b). We should consider strongly nonlinear cases to detect a noticeable departure between the refocused pulse shapes in the linear and nonlinear cases, which appears as a steepening (solid line, Fig. 5b).

## 6. Numerical experiments

In this section we illustrate the results obtained in the previous sections by performing full numerical simulations. We start by describing the numerical set-up for the computer simulations regarding time-reversal in transmission

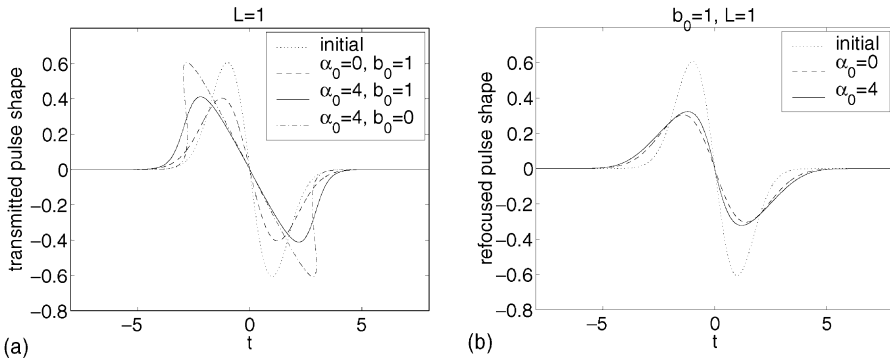


Fig. 4. Transmitted pulse shape  $\tilde{B}_0(L, \cdot)$  (picture a) and refocused pulse shape  $-\tilde{C}_{tr}(\cdot)$  (picture b). The same as in Fig. 3, but we consider here stronger fluctuations  $b_0 = 1$ .

(TRT). The code used is for a three-dimensional shallow water system [6]. For one-dimensional experiments the following reduced (one vertical layer system) is utilized, for which the lateral velocity  $V$  will always be zero.

6.1. Numerical set-up

The shallow water equations used in the numerical simulations are of the form

$$U_t + UU_x + VU_y = -g\eta_x, \tag{84}$$

$$V_t + UV_x + VV_y = -g\eta_y, \tag{85}$$

$$\eta_t + [HU]_x + [HV]_y = 0, \tag{86}$$

where the body of fluid is described by

$$H(x, y, t) = \eta(x, y, t) + h\left(\frac{x}{\varepsilon}, y\right).$$

Since the code is in dimensional variables we control nonlinearity through the amplitude of our initial data and the background (average) depth  $h_0$ . Nonlinearity corresponds to the amplitude-to-depth ratio  $\alpha$ . To normalize the

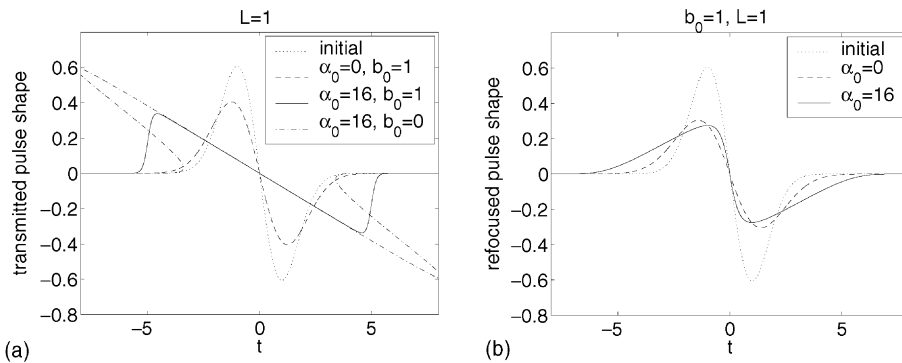


Fig. 5. Transmitted pulse shape  $\tilde{B}_0(L, \cdot)$  (picture a) and refocused pulse shape  $-\tilde{C}_{tr}(\cdot)$  (picture b). The same as in Fig. 3, but we consider here stronger fluctuations  $b_0 = 1$  and a stronger nonlinearity  $\alpha_0 = 16$ .



shallow water speed we choose our parameters so that it is equal to one in the flat regions: namely we set  $g = 1/h_0$  so that  $(gh_0) = 1$ . Several linear validation experiments were presented in [13,20,21] and the numerical solution exhibited very good conservation properties over large propagation distances.

Eqs. (84)–(86) are discretized by an implicit, semi-Lagrangian technique which accomplishes the objective that the stability of the scheme does not depend on the celerity. A fixed staggered grid is defined on the horizontal plane. For the present one-dimensional experiments we have three lines of grid points along the horizontal direction, of which only the middle one is relevant [6,13]. It runs through the middle of the one-dimensional channel. The other two are used for lateral boundary conditions. By a semi-Lagrangian technique we mean that the numerical transport is performed in an Eulerian–Lagrangian manner, relying therefore on the interpolation of the respective grid point-values at the backward-characteristic points of departure. Details are given in Casulli and Cheng [6] and the references within.

Recently this numerical scheme was thoroughly tested in the presence of a random medium and compared to theoretical linear O’Doherty–Anstey results [21]. It was also thoroughly tested for linear time-reversal in reflection [13], where the nonlinear study was initiated. In all simulations the initial pulse is given by the derivative of a Gaussian

$$\eta(x, 0) = U(x, 0) = -10\alpha(x - x_{j0}) \exp\left(\frac{-(x - x_{j0})^2}{0.08}\right) \tag{87}$$

and is centered at node  $x_{j0}$ . Its effective width is equal to one.

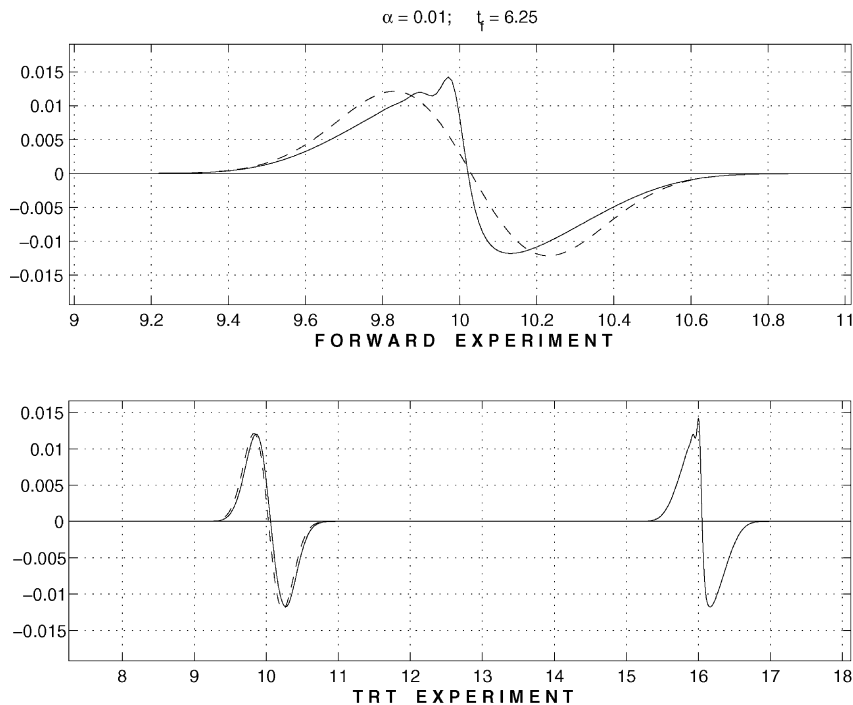


Fig. 6. TRT over a flat channel. (Top) Initial condition (dashed line) and numerical solution (solid line) for  $\alpha = 0.01$ . The critical time is  $t_c = 10$  and the computed time was  $t = 6.25$ . (Bottom) To the right is the time-reversed wave elevation. To the left is the refocused wave (solid line) and the initial profile (dashed line).

### 6.2. Homogeneous nonlinear recompression

Nonlinear 1D acoustic (laboratory) experiments were reported by Tanter et al. [25]. An interesting fact mentioned in [25], concerns the difficulty to carry out laboratory experiments, “for very broadband signals, would require transducers with a huge bandwidth” which makes the experiments a “near impossible task”. Hence the initial condition in [25] was a monochromatic sinusoidal wave. The pulse utilized in our numerical simulations is broadband. The nonlinear mechanism for energy transfer to the higher harmonic components during forward propagation is precisely the point investigated by Tanter et al. [25] in their laboratory experiments. The main goal of their experiments is to check for the reversibility of the energy transfer. The laboratory experiments are carried out for a nonlinear sinusoidal wave propagating in a homogeneous medium. The energy reversibility among harmonics is broken only for propagation longer than the shock formation distance. This shock distance corresponds to the critical time  $t_c$  calculated from Burgers equation. Consider Burgers equation  $u_t + uu_x = 0$ ,  $u(x, 0) = \alpha h(x)$ . Clearly  $t_c = -1/(\alpha h'(x_c))$ . In the simulations we will consider waves of the form

$$h(x) = -10x \exp\left(\frac{-x^2}{0.08}\right).$$

In this case  $t_c = 1/(10\alpha)$ , corresponding to the point with the largest negative slope ( $x_c = 0$ ).

In our first experiment we illustrate nonlinear time reversal in a homogeneous medium. As will be shown this amounts to energy recompression in Fourier space, namely an inverse cascading of energy from higher wavenumbers up to lower wavenumbers.

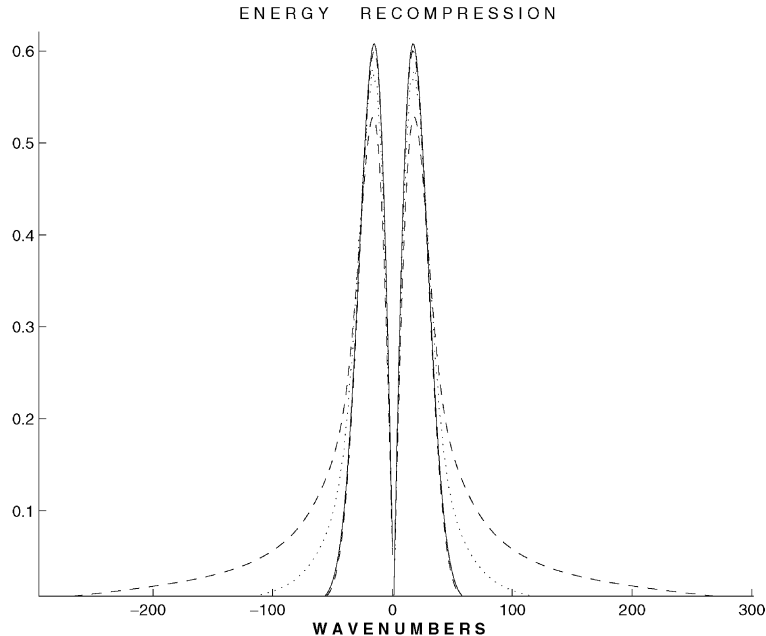


Fig. 7. At the center (solid line) we have the absolute value of the Fourier amplitudes for the initial data:  $|\widehat{\eta}(x, 0)|$ . The generation of higher harmonics is depicted by the dotted line, which represents the spectrum at  $t = 3.0$ , and by the dashed line (at the final time  $t = 6.25$ ). The Fourier spectrum of the time-reversed data is the wider dashed line and the spectrum of the refocused pulse is represented by the dashed line coinciding with the solid line.

We solve the nonlinear shallow water system (84–86) in a flat channel (i.e., a homogeneous medium) with initial condition (87) ( $\alpha = 0.01$ ;  $x_{j0} = 10$ ). At critical time  $t_c = 10$  a shock will form. With  $\Delta x = 0.01$  and  $\Delta t = 0.0025$  we are able to propagate numerically our pulse all the way up to  $t = 6.25$  before spurious oscillations arise due to the mesh coarsening along the steep front. Increasing the resolution we can go further in time. The steep front (at time  $t = 6.25$ ) is presented in the top part of Fig. 6 where it was shifted for comparison with the initial profile (centered at  $x_{j0}$ ). The steep profile is time-reversed, propagated backward and it “refocuses” at the origin  $x_{j0}$  as seen at the bottom part of Fig. 6. There is no inhomogeneity and therefore no multiple scattering. Here refocusing (or recompression) should be understood in Fourier space. In Fig. 7 we can see the cascading of energy in Fourier space through the generation of higher harmonics due to nonlinear interaction. At the end of the forward simulation the Fourier spectrum is about five times broader than that at  $t = 0$  (solid line). After time reversion the back propagated pulse is such that the Fourier spectrum recompresses back to its narrower band (dashed line coinciding with solid line). This experiment is of the same nature as in [25] but for a broadband initial data.

### 6.3. The O’Doherty–Anstey theory and apparently viscous shocks

Now we consider inhomogeneous (disordered) media. The influence of the topography is felt through the depth function  $H(x, y, t)$ . The disordered medium is constructed by sampling a given number of random heights which

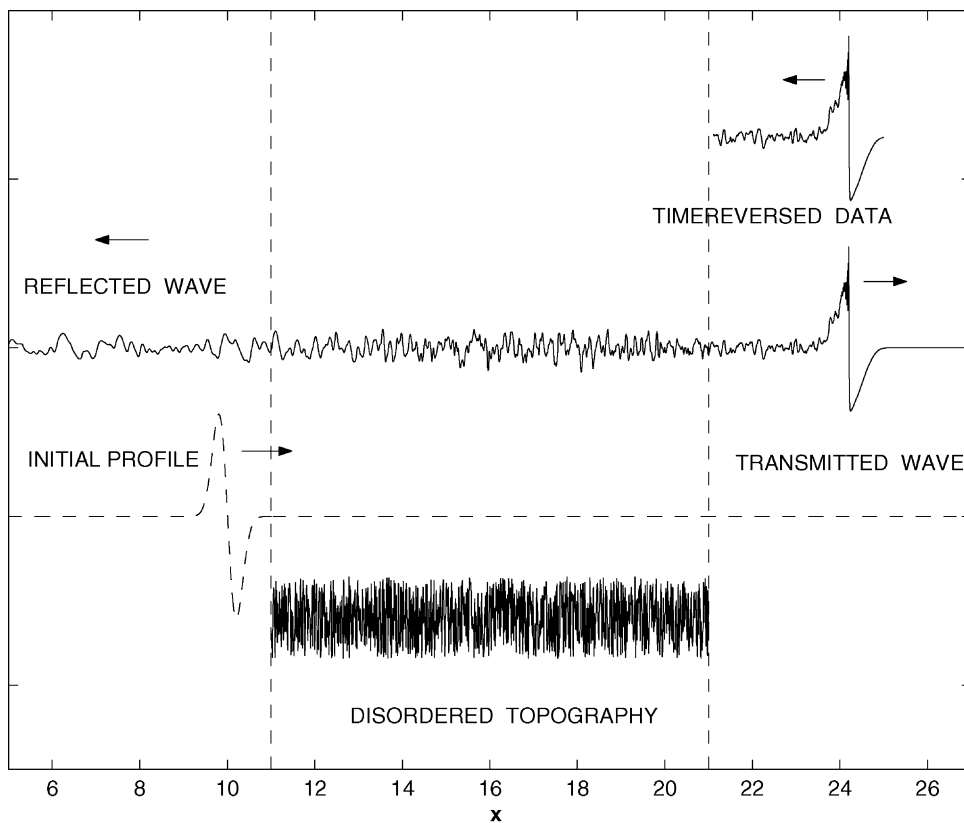


Fig. 8. The set-up for the numerical simulations.

are allocated at the one-dimensional mesh points. The correlation length of the disordered medium is represented by  $\epsilon = m\Delta x$ ,  $m \geq 1$ , in the simulations. These random heights are then connected by straight lines to form the synthesized topography. The channel has a total length of  $L$  units while the disordered topography covers only part of it (cf. Fig. 8). In the following experiments we use the mesh  $\Delta x = 0.0025$ ,  $\Delta y = 0.1$  and  $\Delta t = 0.000625$ . These discretization parameters were chosen in order to get the desired accuracy and proper physical behavior when the nonlinear pulse interacts with the rapidly varying random medium ( $\epsilon = 0.01$ ). Note that in this case  $m = 4$ . The fluctuations are 60% of the average depth in all experiments throughout this paper. These values were adjusted so that we could capture the desired features of the analysis, such as the apparent viscosity of the effective Burgers equation in Section 4. In particular we want to capture the viscous shock profile. They were not chosen due to stability requirements. We set  $\alpha = 0.004$  through the initial data which leads to  $t_c = 25$ . We chose this value so that we can propagate the inviscid Burgers solution for a longer time interval (than in the previous example) and compare it with its corresponding viscous profile as presented in Fig. 9. The waves have propagated about 18 units and the inviscid profile is steep. We clearly see the viscous shock due to the stochastic forcing.

To emphasize this viscous shock (stochastic) mechanism the evolution of the nonlinear wavefront is observed at different times. They are centered about the solution at  $t = 6.25$  in Fig. 10. Over a flat bottom the pulse continuously steepens under the “Burgers-like” nonlinearity (bottom of Fig. 10). In the presence of the stochastic forcing, promoted by the disordered topography, the dynamics is effectively governed by a viscous Burgers equation of the form  $u_t + uu_x = \nu u_{xx}$ ,  $u(x, 0) = \alpha h(x)$  as presented earlier. A complete analysis of the viscous shock structure is presented in Whitham [26] (Chapter 4; in particular Section 4.4). The

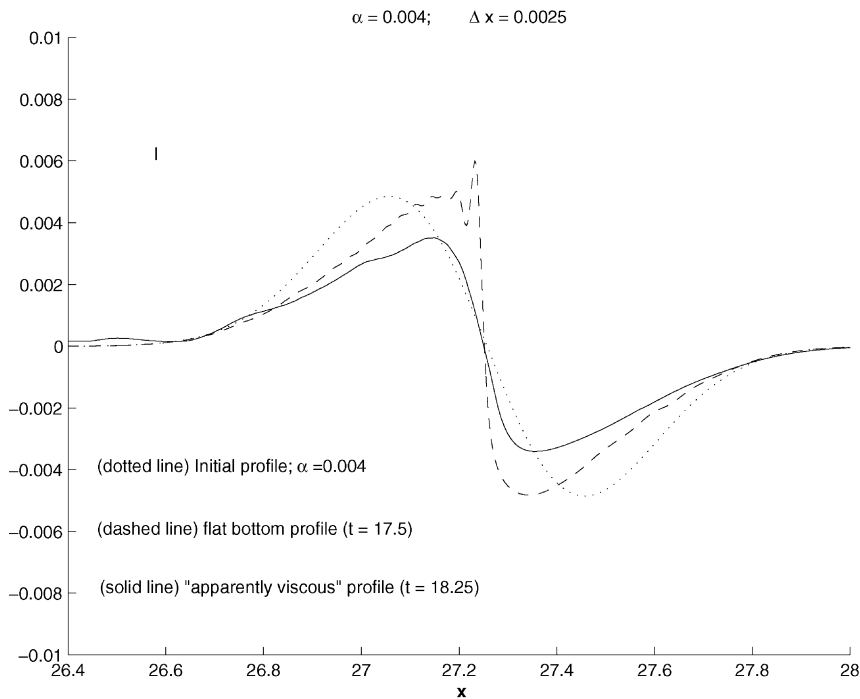


Fig. 9. The dashed line represents the inviscid (flat bottom) solution and the solid line the “apparently viscous” solution in the presence of a disordered topography. The inviscid solution was slightly shifted for a better comparison of the slopes. The dotted line represents the initial profile.

shock structure scales according to the  $R^{-1}$ , where this Reynolds number is defined as  $R = \alpha/(2\nu)$ . Moreover after the viscous shock reaches its limiting  $O(R^{-1})$  transition layer a traveling wave arises as indicated in Whitham [26] (Section 4.3). At the top of Fig. 10 the traveling wavefront regime (effectively speaking) can be clearly observed. The wavefront ceases to steepen and saturates on a given scale. We have verified numerically that for increasing values of  $\alpha$  the front saturates at a smaller  $R^{-1}$  value, while increasing the  $\epsilon$  of the disorder the slope becomes milder: larger correlation lengths imply in larger effective viscosity  $\nu$ .

#### 6.4. Nonlinear time reversal in transmission

In this section, we illustrate time-reversal in transmission for a pulse that has propagated beyond its critical distance. We let  $\alpha = 0.01$  which implies that  $t_c = 10$ . In Fig. 11 we have the initial profile to the left and two time-reversed data to our right (shifted above the free surface for better visualization). The time-reversed profiles were recorded at  $t = 13.75$  and  $t = 15$  both greater than  $t_c$ . As discussed above we have a viscous shock steeper than those presented in Fig. 10 because nonlinearity has been increased. But we are not in the situation described by Tanter et al. [25] because our wavefront developed into a viscous shock structure and not into a discontinuity, as in [25]. The time-reversed profiles are back-propagated into the disordered medium and refocus at the origin  $x_{j0} = 10$  as expected. We can not tell the difference between the two refocused profiles.

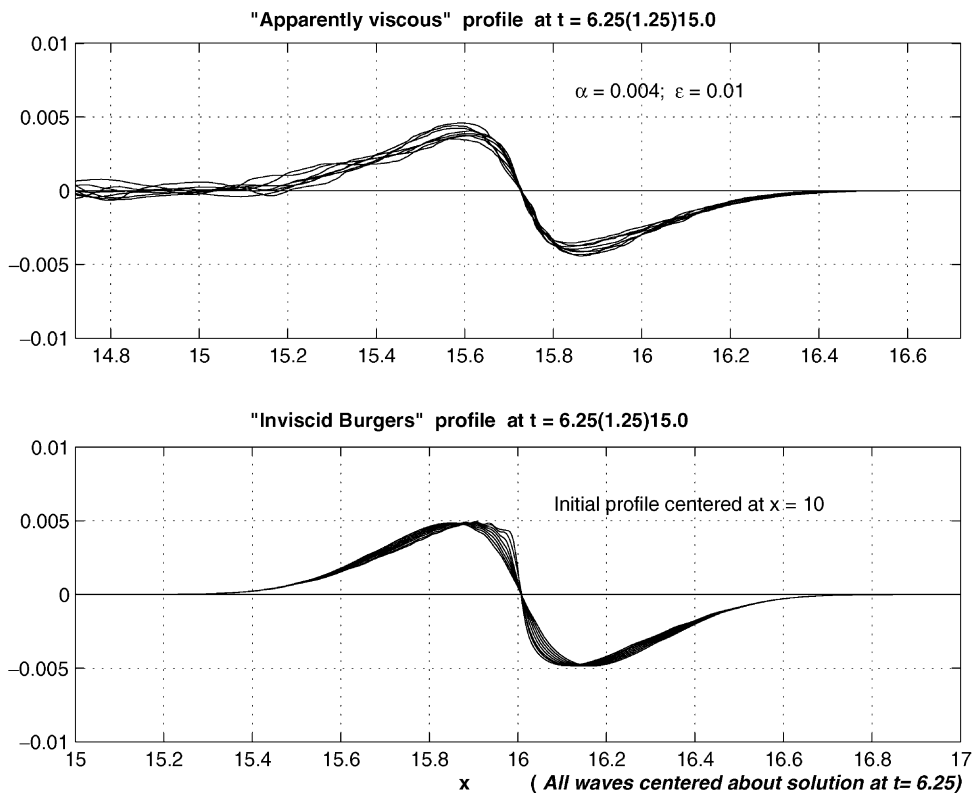


Fig. 10. "Apparently viscous" profile (top) and inviscid profile (bottom) at eight different instants in time (with intervals equal to 1.25), all centered about the solution at  $t = 6.25$ .

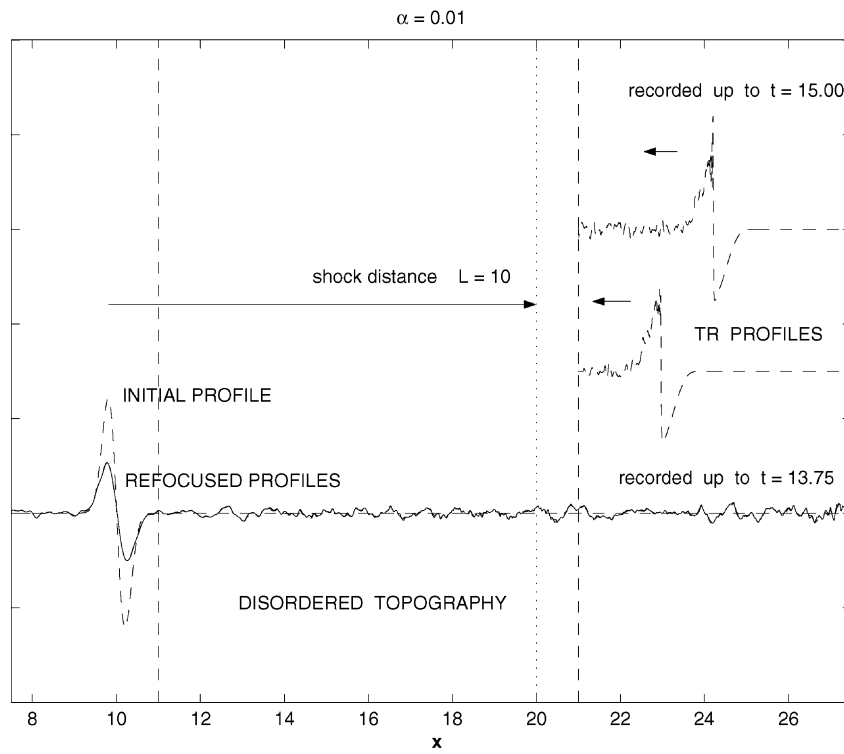


Fig. 11. Refocusing beyond the shock distance (vertical dotted line). The initial profile (dashed line) is to the left and two time-reversed data are to the right (shifted above the free surface for better visualization). The time-reversed profiles are back-propagated into the disordered medium and refocus at the origin  $x_{j_0} = 10$  at  $t = 13.75$  and  $t = 15$ , respectively.

## 7. Conclusion

In this paper, we have addressed the propagation of nonlinear water waves in a disordered one-dimensional topography. In the presence of properly scaled stochastic forcing the solution is regularized leading to a viscous shock profile which depends on the degree of nonlinearity and on the power spectral density of the random fluctuations of the bottom. We have actually shown that the transmitted wave is governed by an effective viscous Burgers equation. We have also considered time-reversal problems and shown that the regularization effect allows us to extend the reversibility of the dynamics beyond the shock critical time.

We believe that these results can be extended to other nonlinear systems. For instance, the gas dynamics equation, or the nonlinear acoustic equation [25], can be written directly from the shallow water equations if we consider a polytropic gas with adiabatic exponent equal to two [26]. We believe that nonlinear systems that support Riemann invariants are likely to behave qualitatively like the model addressed in our paper.

Another type of generalization would consist in addressing configurations with spatial dimension higher than one, such as the Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation [27,15]. We think that the shock regularization due to random fluctuations should survive in higher dimensions. However, we do not expect that time reversal experiments will be efficient in nonlinear 2D or 3D media. Indeed we should address practical configurations where the aperture of the mirror is small. In such a configuration, a very small part of the total signal is recorded, and the nonlinearity during the time-reversal experiment will be very different.

## Acknowledgements

This work was supported by the MGSS program while the authors were visiting Stanford University and by ONR under grant N00014-02-1-0089. The authors would like to thank Vincenzo Casulli (Università di Trento, Italy) for the copy of his code (TRIM3D). Josselin Garnier acknowledges support from the program ACI-NIM-2003-94. André Nachbin acknowledges the support by CNPq/Brazil under grant 300368/96-8.

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