Transmission of matter-wave solitons through nonlinear traps and barriers

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The transmissions of matter-wave solitons through linear and nonlinear inhomogeneities induced by the spatial variations of the trap and the scattering length in Bose-Einstein condensates are investigated. The enhanced transmission of a soliton through a linear trap by a modulation of the scattering length, is exhibited. The theory is based on the perturbed inverse scattering transform for solitons, and we show that radiation effects are important. Numerical simulations of the Gross-Pitaevskii equation confirm the theoretical predictions.

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I. INTRODUCTION

The transmission of matter wave packets through inhomogeneities of different types of Bose-Einstein condensates (BECs) has recently attracted a lot of attention, because this phenomenon is important for the design of control methods of the soliton parameters and atomic soliton lasers [1]. The transmission and reflection of bright and dark matter wave solitons has been studied in the case of *linear* inhomogeneities, induced by the variations in space of the potential field [2–8]. In particular, the effect of a potential step or impurity, including the soliton train evolution, has been analyzed in Ref. [6]. The adiabatic dynamics of a dark soliton, as well as the radiative wave emission leading to the dark soliton degradation, has been studied in Ref. [7]. Finally, the continuous wave emission by a bright soliton in an optical lattice has been addressed in Ref. [8]. The case of inhomogeneities produced by spatial variations of the scattering length has been less investigated. In Refs. [9–11] the variational approach has been applied and numerical simulations have been performed. When the linear and nonlinear inhomogeneities compete with each other, direct numerical simulations of the soliton propagation have been carried out. The enhanced soliton transmission through a linear barrier is observed for well-chosen parameters of the nonlinear potential [12]. The explanation of this phenomenon, as shown in this paper, requires to take into account the radiative effects when the soliton interacts with the nonlinear potential.

The purpose of this work is to develop the theory describing the transmission of matter wave solitons through nonlinear barriers and traps. Such barriers can be produced by using the Feshbach resonance method, namely by the local variation of the external magnetic field $B(z)$ in space near the resonant value B_c [13]. By the small variation of the field near the resonant value we can induce the large variations of the scattering length in space according to the formula

$$a_s(z) = a_b\left(1 - \frac{\Delta}{B_c - B(z)}\right),$$

where a_b is the background value of the atomic scattering length and Δ is the resonance width. Optical methods for manipulating the value of the scattering length are also possible [14]. As a result, the mean field nonlinear coefficient (which is proportional to the scattering length a_s) in the Gross-Pitaevskii equation has a spatial dependence. We will use the perturbed inverse scattering transform theory (see, for example, Refs. [15–17]) to describe the transmission of bright matter wave solitons through the nonlinear barriers. This approach allows us to analyze the adiabatic dynamics of solitons as well as the radiative processes during the soliton propagation through inhomogeneities.

II. THE MODEL

The quasi-one-dimensional Gross-Pitaevskii (GP) equation describing the wave function of BEC in an elongated trap has the form [18]

$$i\hbar\,\psi_\tau + \frac{\hbar^2}{2m}\psi_{zz} - V(z)\psi - g_{1\mathrm{D}}(z)|\psi|^2\psi = 0. \quad (1)$$

Here $g_{1\mathrm{D}}(z)=2\hbar\,\omega_\perp a_s(z)$, where $\omega_\perp$ is the transverse oscillator frequency, $a_s(z)$ is the spatially dependent atomic scattering length, and $V(z)$ is the linear potential. Both $a_s(z)$ and $V(z)$ are assumed to be constant outside a given domain, where we assume that the scattering length takes the constant negative value a_{s0} and V is zero, $\int|\psi|^2dz=N$, where N is the number of atoms. We denote by $\bar{g}_{1\mathrm{D}}=2\hbar\,\omega_\perp a_{s0}$ the reference value of the nonlinear coefficient.

We first rewrite the GP equation in dimensionless variables. We introduce the healing length $z_0=\hbar/\sqrt{n_0\bar{g}_{1\mathrm{D}}m}$ and the corresponding time $\tau_0=2z_0/c$, where $c=\sqrt{n_0\bar{g}_{1\mathrm{D}}/m}$ is the Bogoliubov speed of sound and n_0 the peak density. Denoting $x=z/z_0$, $t=\tau/\tau_0$, and $u=\psi/\sqrt{n_0}$, the normalized mean field wave function u satisfies the dimensionless GP equation in the form of the nonlinear Schrödinger (NLS) equation

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$$iu_t + u_{xx} + 2|u|^2 u = V_{nl}(x)|u|^2 u + V_l(x)u, \tag{2}$$

with $V_{nl}(x)=2-2a_s(z_0x)/a_{s0}$ and $V_l(x)=\tau_0 V(z_0x)/\hbar$. If $V_l=V_{nl}=0$, then this equation can be reduced to the standard NLS equation that supports soliton solutions. The bright soliton solution is

$$u_s(x,t) = 2\nu \operatorname{sech}\{2\nu[x - x_s(t)]\}\exp\{2i\mu[x - x_s(t)] + i\phi_s(t)\}.$$

The soliton amplitude is 2ν and its velocity is 4μ. The soliton center and phase $x_s(t)$ and $\phi_s(t)$ satisfy

$$\frac{dx_s}{dt} = 4\mu, \quad \frac{d\phi_s}{dt} = 4(\nu^2 + \mu^2).$$

The matter-wave soliton moving in the linear and nonlinear potentials V_l and V_{nl} experiences velocity and mass modulations and emits radiation. To describe this process we use the perturbation theory based on the inverse scattering transform (IST). The IST is a linearization of the NLS equation based on the fact that u can be characterized by a set of spectral data for a linear operator in which u plays the role of a potential [19]. The matter wave can be decomposed as the sum of localized soliton parts (associated to the discrete eigenvalues) and delocalized radiation (associated to the continuous spectrum). The perturbed IST for the perturbed NLS equation (2) describes the evolutions of the localized soliton part of the matter wave and the delocalized radiation in a complete but complex way [15]. A tractable perturbation analysis can be carried out by using series expansions with respect to the amplitudes V_{l0} and V_{nl0} of the linear and nonlinear dimensionless potentials. In physical variables, $V_{nl0}=2\ \sup_x|a_s(x)/a_{s0}-1|$ and $V_{l0}=\tau_0\ \sup_x|V(x)|/\hbar$.

III. QUASIPARTICLE APPROACH

Applying the first-order perturbed IST theory [15], we obtain the system of equations for the soliton amplitude and velocity

$$\frac{d\nu}{dt} = 0, \quad \frac{d\mu}{dt} = -\frac{1}{4\nu}W'(\nu,x_s), \quad \frac{dx_s}{dt} = 4\mu,$$

where the prime stands for a derivative with respect to x and the effective potential has the form

$$W(\nu,x) = W_l(\nu,x) + W_{nl}(\nu,x),$$

with

$$W_l(\nu,x) = \nu\int_{-\infty}^{\infty} \frac{1}{\cosh^2(z)} V_l\left(\frac{z}{2\nu} + x\right)dz, \tag{3}$$

$$W_{nl}(\nu,x) = 2\nu^3\int_{-\infty}^{\infty} \frac{1}{\cosh^4(z)} V_{nl}\left(\frac{z}{2\nu} + x\right)dz. \tag{4}$$

In this first approximation terms of order V_{l0}^2 and V_{nl0}^2 are neglected. We can thus write the effective equation describing the dynamics of the soliton center as a quasiparticle moving in the effective potential W,

$$\nu_0\frac{d^2x_s}{dt^2} = -W'(\nu_0,x_s), \tag{5}$$

where $4\nu_0$ is the mass (number of atoms) of the incoming soliton. This system has the integral of motion

$$\frac{\nu_0}{2}\left(\frac{dx_s}{dt}\right)^2 + W(\nu_0,x_s) = 8\nu_0\mu_0^2, \tag{6}$$

where $4\mu_0$ is the velocity of the incoming soliton. Note that this approach gives the same result as the adiabatic perturbation theory for solitons that is a first-order method as well. This adiabatic perturbation theory was originally introduced for optical solitons [20] and it was recently applied to matter-wave solitons [21]. In this work, it is the first step as we will include second-order and radiation effects in the next section. Let us briefly discuss the main results that can be obtained with the quasiparticle approach.

Barrier potential: Let us first examine the case where the potential V is a barrier, meaning that $V\geq 0$ and $\lim_{|x|\to\infty} V(x)=0$. When the soliton approaches the barrier, it slows down, and it eventually goes through the barrier if its input energy is above the maximal energy barrier, meaning $8\nu_0\mu_0^2 > W_{\max}(\nu_0)$. For a linear barrier potential which is an even function, we have $W_{\max}(\nu)=W_l(\nu,0)$, where W_l is given by (3). For a nonlinear barrier potential which is an even function, we have $W_{\max}(\nu)=W_{nl}(\nu,0)$. After passing through the barrier, the soliton recovers its initial mass and velocity. In that sense, the transmission coefficient is one.

If, on the contrary, the velocity of the incoming velocity is such that $8\nu_0\mu_0^2 < W_{\max}(\nu_0)$, then the soliton is reflected by the barrier. After the interaction with the barrier, the soliton velocity takes the value $-4\mu_0$. In that sense, the transmission coefficient is zero.

Trap potential: We now examine the case where the potential is a trap, meaning that $V\leq 0$ and $\lim_{|x|\to\infty} V(x)=0$. When the soliton approaches the trap, it speeds up, and it eventually goes through the barrier whatever its initial velocity is. This is the prediction of the quasiparticle approach. However, we shall see that the interaction with the trap generates radiation and reduces the mass and energy of the soliton. As a result, the soliton may not be able to escape the trap if its initial velocity is too small. We shall discuss this point in the next section.

IV. RADIATION EFFECTS

The properties of the radiation emitted by the soliton interacting with the potentials V_l and V_{nl} are determined by the Jost coefficients $a(t,\lambda)$ and $b(t,\lambda)$ of the continuous spectrum of the associated linear spectral problem [15–17]. We assume that the linear and nonlinear potentials are localized functions, and we denote by $\hat{V}(k)=\int V(x)\exp(ikx)dx$ their Fourier transforms. If the soliton goes through the potential, then we find

$$\frac{b}{a}(t=+\infty,\lambda)=\frac{-i\pi(\lambda-\mu-i\nu)^2}{16\mu^3\cosh\left(\frac{\pi}{4}\frac{\lambda^2+\nu^2-\mu^2}{\mu\nu}\right)}\left(\hat{V}_l(k(\lambda,\nu,\mu))+\frac{[(\lambda+\mu)^2+\nu^2][(\lambda-3\mu)^2+8\mu^2+\nu^2]}{12\mu^2}\hat{V}_{nl}(k(\lambda,\nu,\mu))\right),$$

where

$$k(\lambda,\nu,\mu)=\frac{(\lambda-\mu)^2+\nu^2}{\mu}.$$

This formula is obtained from the first-order perturbation IST theory [15] and it allows us to capture the second-order evolution of the soliton parameters, as we show now. The total mass (number of atoms) and energy (Hamiltonian) are preserved by the perturbed NLS equation (2),

$$\mathcal{N}=\int_{-\infty}^{\infty}|u|^2dx,$$

$$\mathcal{H}=\int_{-\infty}^{\infty}\left[|u_x|^2-|u|^4+V_l(x)|u|^2+\frac{1}{2}V_{nl}(x)|u|^4\right]dx.$$

The total mass and energy have contributions from the radiation and from the soliton. The idea is to compute the radiated mass and energy, and to use the conservation of the total mass and energy to derive the decay of the soliton mass and energy, which in turn gives the decay of the soliton parameters (ν,μ). As shown in Ref. [19], the radiated mass density is given by $n(\lambda)=\ln[1+|b/a(\lambda)|^2]/\pi$. It is therefore proportional to V_{l0}^2 and V_{nl0}^2, so we can use the second-order approximation

$$n(\lambda)\simeq\frac{1}{\pi}\left|\frac{b}{a}(t=+\infty,\lambda)\right|^2.$$

The total mass and energy can be expressed in terms of the radiation and soliton components as

$$\mathcal{N}=4\nu+\int_{-\infty}^{\infty}n(\lambda)d\lambda,$$

$$\mathcal{H}=16\nu\mu^2-\frac{16}{3}\nu^3+2W(\nu,x_s)+4\int_{-\infty}^{\infty}\lambda^2n(\lambda)d\lambda.$$

Note that this expression of the energy $\mathcal{H}$ is valid up to second order and it generalizes the first-order expression (6). Therefore, the emission of radiation involves a decay of the soliton mass and energy which is proportional to V_{l0}^2 and V_{nl0}^2. The coefficients (ν_T,μ_T) of the transmitted soliton are

$$\nu_T=\nu_0-\frac{1}{4}\int_{-\infty}^{\infty}n(\lambda)d\lambda, \tag{7}$$

$$\mu_T=\mu_0-\frac{1}{8}\int_{-\infty}^{\infty}\left(\frac{\lambda^2}{\mu_0\nu_0}+\frac{\nu_0}{\mu_0}-\frac{\mu_0}{\nu_0}\right)n(\lambda)d\lambda. \tag{8}$$

These formulas allow us to study and characterize the transmission of a soliton through a general barrier in various regimes. We can consider the transmission and/or reflection of a bright soliton through a nonlinear barrier, the transmission and/or trapping in a nonlinear trap, and competition effects for the transmission through the superposition of nonlinear and linear potentials.

In the next sections, we compare our theoretical predictions with results from numerical simulations of the one-dimensional GP equation, with a Gaussian linear potential and/or a nonlinear Gaussian potential,

$$V_l(x)=V_{l0}\exp\left(-\frac{x^2}{x_c^2}\right),\quad V_{nl}(x)=V_{nl0}\exp\left(-\frac{x^2}{x_c^2}\right).$$

We consider the transmission of a soliton incoming from the left homogeneous half-space with parameters (ν_0,μ_0). We plot the variations of the soliton parameters versus the value of the velocity for $\nu_0=0.5$ and $x_c=0.5$. We consider different combinations of linear and nonlinear potentials.

Nonlinear barrier: We consider the case $V_{l0}=0$ and $V_{nl0}>0$. We have seen in Sec. III that the condition for the transmission through a nonlinear barrier is that the velocity of the incoming velocity is large enough so that $8\nu_0\mu_0^2>W_{\max}(\nu_0)$. Note that this means that the critical velocity parameter μ_{crit}, defined by the identity $8\nu_0\mu_{\text{crit}}^2=W_{\max}(\nu_0)$, is of the order of $V_{nl0}^{1/2}$ for $V_{nl0}\ll 1$. If the transmission condition fails, then the soliton is reflected.

These results are obtained in the quasiparticle approach and neglect the radiation emission phenomenon. Taking into account radiation yields that the transmission is not complete in the case $8\nu_0\mu_0^2>W_{\max}(\nu_0)$, in the sense that the transmitted soliton mass is not equal to the incoming soliton mass (see Fig. 1). The mass loss is described by (7), and it is of the order of V_{nl0} for $V_{nl0}\ll 1$.

Nonlinear trap: We consider the case $V_{l0}=0$ and $V_{nl0}<0$. The quasiparticle approach predicts full soliton transmission, but neglects radiation phenomena. When taking into account radiation emission, we can exhibit the mass and energy loss during the interaction with the trap potential. The losses are described by (7) and (8) and are accurate for an initial velocity large enough. Indeed, if the initial velocity is not large enough, then the energy loss during the interaction with the potential does not allow the soliton to escape the trap.

For small initial velocity, the expression for the radiation is not precise enough, because the velocity experiences a modulation which is relatively large compared to its initial

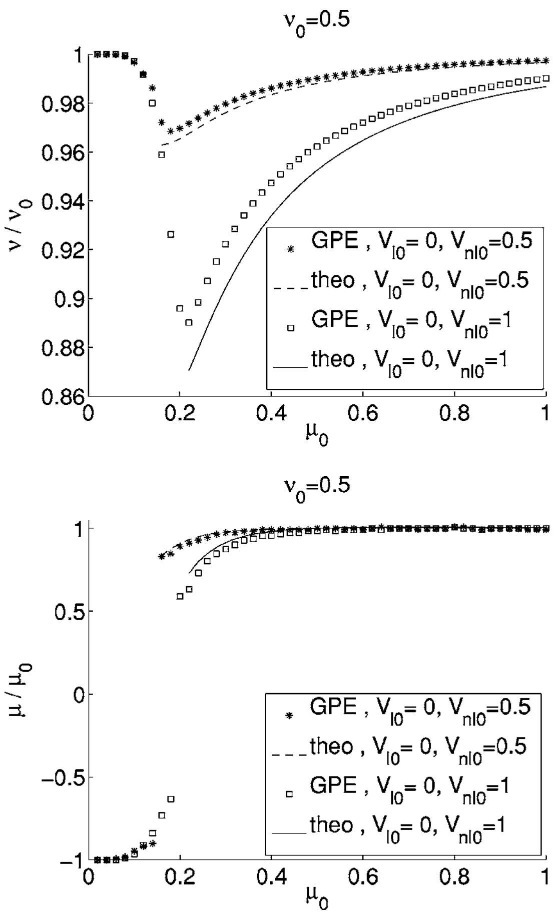

FIG. 1. Soliton parameters variations versus input velocity parameter μ_0 for a nonlinear barrier. The theoretical reflection condition is $8\nu_0\mu_0^2 < W_{\max}(\nu_0)$ which gives $\mu_{\text{crit}}=0.21$ for $V_{nl0}=1$ and $\mu_{\text{crit}}=0.15$ for $V_{nl0}=0.5$, in excellent agreement with the simulations. The predicted mass and velocity reductions are also quantitatively accurate, especially for $V_{nl0}=0.5$.

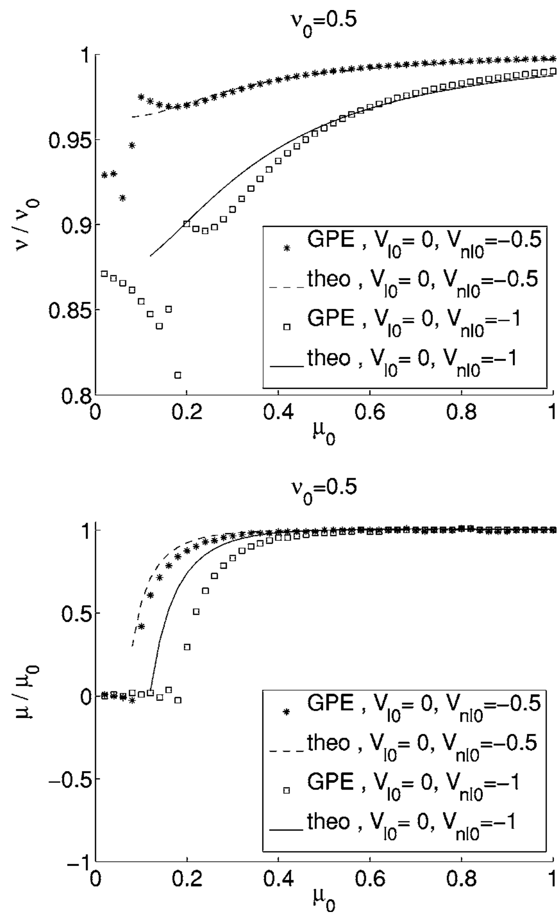

FIG. 2. Soliton parameters variations versus input velocity parameter μ_0 for a nonlinear trap. The theoretical trap condition is $\mu_T < 0$ which gives $\mu_{\text{crit}}=0.12$ for $V_{nl0}=-1$ and $\mu_{\text{crit}}=0.07$ for $V_{nl0}=-0.5$. The agreement with the simulations is noticeable for $V_{nl0}=-0.5$, while the case $V_{nl0}=-1$ is only in qualitative agreement with the simulations.

value. When the soliton center is x, the soliton velocity is given by $4\mu(x)$ with

$$\mu(x) = \sqrt{\mu_0^2 - \frac{W_{nl}(\nu_0,x)}{16\nu_0}}.$$

Its maximal value is $\mu_{\max}=\sqrt{\mu_0^2 - W_{nl}(\nu_0,0)/(16\nu_0)}$ where $W_{nl}(\nu_0,x)<0$ is given by (4). During the interaction with the barrier, the soliton velocity is about $4\mu_{\max}$, so the energy loss can be estimated by $-4\int \lambda^2 n(\lambda) d\lambda$, where $\mu_{\max}$ is substituted for μ_0 in the expression of b/a. If a negative value μ_T is obtained in (8), then this means that the soliton has not been transmitted, but was trapped by the nonlinear potential (see Fig. 2). Note that $n(\lambda) \sim V_{nl0}^2$ so that the critical value μ_{crit} for trapping is of the order of V_{nl0}.

If the initial velocity is large enough to ensure transmission, then the soliton emits radiation and loses mass. The mass loss is described by (7) and it is of the order of V_{nl0} for $V_{nl0} \ll 1$.

Enhanced transmission by nonlinear modulation: As pointed out in Ref. [21], the nonlinear potential V_{nl} can help a soliton going through a trap potential V_l. We illustrate this assertion based on numerical experiments in this section and justify it with our perturbed IST approach.

By comparing the transmission through a linear trap in presence or in absence of a nonlinear positive potential, we confirm the numerical conjecture that the transmission coefficient can be significantly increased by a nonlinear modulation (see Figs. 3 and 4). In fact, the radiation emitted by the soliton due to the interaction with the linear trap and with the nonlinear potential can cancel each other, resulting in an enhanced soliton transmittivity.

A similar result can be obtained with a linear barrier. A nonlinear negative modulation can help the soliton going through the linear barrier by reducing the radiation emission and by reducing the minimal velocity reached by the soliton during the interaction. This was predicted by numerical simulations in Ref. [12].

For consistency we propose an experimental configuration where the enhanced transmission could be observed for the ^{7}Li condensate. The transverse frequency $\omega_\perp \approx 2\pi \times 10^3$ Hz and the density is $n=10^9$ m^{-1}. The healing length and speed of sound are $z_0 \simeq 2$ μm and $c \approx 5$ mm/s, respectively. As a typical experiment, we could consider a soliton with

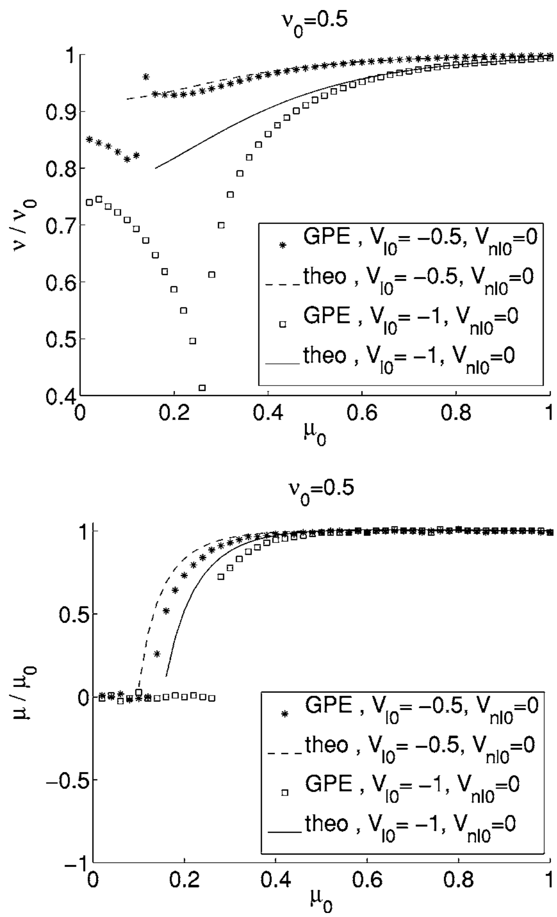

FIG. 3. Soliton parameters variations versus input velocity parameter μ_0 for a linear trap. The theoretical trap condition is $\mu_T<0$ which gives $\mu_{\text{crit}}=0.15$ for $V_{l0}=-1$ and $\mu_{\text{crit}}=0.1$ for $V_{l0}=-0.5$, in good agreement with the simulations in the case $V_{l0}=-0.5$.

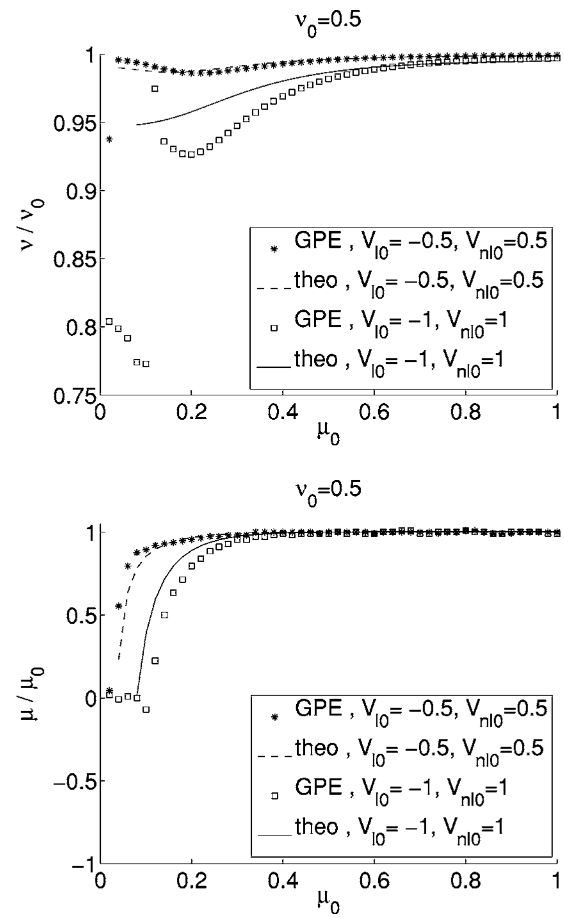

FIG. 4. Soliton parameters variations versus input velocity parameter μ_0 for a linear trap superposed with a nonlinear barrier. The theoretical trap condition is $\mu_T<0$ which gives $\mu_{\text{crit}}=0.08$ for $V_{nl0}=1$, $V_{l0}=-1$ and $\mu_{\text{crit}}=0.04$ for $V_{nl0}=0.5$, $V_{l0}=-0.5$, in good agreement with the simulations in the case $V_{nl0}=0.5$, $V_{l0}=-0.5$.

about 10^3 atoms and width $\approx 2z_0 \approx 4$ μm (so that $\nu=0.5$) and a linear trap with Gaussian shape, normalized amplitude 0.5 and width $\approx z_0 \approx 2$ μm (so that $x_c=0.5$). The theoretical prediction is that such a soliton is trapped if its velocity is below $0.55c$ (because $\mu_{\text{crit}} \simeq 0.14$, see Fig. 3). However, in this experiment, we can consider variations of the external magnetic field B around the value 352 G, where the scattering length has the minimal value ≈ -0.23 nm. Increasing the field to the value $B=450$ G we can increase the scattering length to the value ≈ -0.18 nm. This means that the scattering length can be varied by 25%, and thus a nonlinear barrier V_{nl} with normalized amplitude 0.5 can be generated. The theoretical prediction is that, in the presence of the linear trap and the nonlinear barrier, the soliton will be trapped only if its velocity is below $0.2c$ (because $\mu_{\text{crit}} \simeq 0.05$, see Fig. 4), and it will be transmitted otherwise. This means that the nonlinear modulation dramatically enhances the domain of parameters for which solitons can be transmitted through the linear trap.

V. CONCLUSION

In this paper we have investigated the time-dependent nonlinear scattering of bright solitonic matter waves through different types of barriers. We have considered the transmission of matter waves through inhomogeneities in the form of localized linear and nonlinear potentials. The adiabatic dynamics of the wave packets as well as the radiative processes during the transmission have been analyzed.

To analyze the dynamics we use the perturbed IST theory, which allows us to predict the trapping of a bright soliton by a trap potential and the reflection of a soliton by a barrier potential. The parameters (mass and velocity) of the transmitted soliton can be estimated by computing the radiated mass and energy and by using the conservations of the total mass and energy. The enhanced transmission of a soliton through a linear trap by a nonlinear modulation of the scattering length is explained by this theory.

The analytical predictions have been checked by comparisons with numerical simulations of the GP equation. The formulas are valid in the asymptotic framework where the potentials have small amplitudes. It turns out that, for small or moderate potential amplitudes $|V_{l0}| \leq 0.5$, $|V_{nl0}| \leq 0.5$, the perturbed IST theory gives quantitatively accurate predictions. For large potential amplitudes $|V_{l0}| \geq 1$, $|V_{nl0}| \geq 1$, the theoretical predictions of the perturbed IST theory are still in qualitative agreement with the numerical results. This means that the perturbed IST theory is useful for probing the parameter space and exhibiting interesting phenomena. The enhanced transmission can be observed in the experiments with bright matter wave solitons in elongated trap with proper variation in space of the external magnetic field and the trap potential [21–23]. One of the problems that should be addressed for future consideration by this approach is the nonlinear resonant scattering on the (periodic or random) chain of nonlinear barriers [24] and the scattering on time-dependent linear and nonlinear barriers [25]. The latter problem is important for many areas of condensed matter.

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