

Identification of Green's Functions Singularities by Cross Correlation of Noisy Signals

Claude Bardos¹, Josselin Garnier², and George Papanicolaou³

¹ Laboratoire Jacques-Louis Lions, Université Paris VII, 2 Place Jussieu, 75251 Paris Cedex 5, France

² Laboratoire de Probabilités et Modèles Aléatoires & Laboratoire Jacques-Louis Lions, Université Paris VII, 2 Place Jussieu, 75251 Paris Cedex 5, France

³ Mathematics Department, Stanford University, Stanford, CA 94305

E-mail: bardos@math.jussieu.fr, garnier@math.jussieu.fr, papanico@math.stanford.edu

Abstract. In this paper we consider the problem of estimating the singular support of the Green's function of the wave equation in a bounded region by cross correlating noisy signals. A collection of sources with unknown spatial distribution emit stationary random signals into the medium, which are recorded at two observation points. We show that the cross correlation of these signals has enough information to identify the singular component of the Green's function, which provides an estimate of the travel time between the two observation points. As in the recent work of Y. Colin de Verdière [math-ph/0610043], we use semiclassical arguments to approximate the wave dynamics by classical dynamics. Next we use the ergodicity of the ray dynamics to obtain an explicit expression of the cross correlation of the noisy signals. We also show that this approach is statistically stable when the averaging time is long enough, and that the accuracy of the travel time estimation is directly related to the spatial correlation function of the sources.

PACS numbers: 86A15,81Q20,37A25,60G60

Submitted to: *Inverse Problems*

1. Introduction

The analysis of imaging techniques in complex media is motivated by many applications [4], in geophysics in particular [18]. In this paper, we consider the problem of estimating the Green's function by cross correlating noisy signals. We assume that spatially distributed sources with an unknown density emit stationary random signals into the medium, which are recorded at two observation points. We analyze the cross correlation function of these signals and show that, under certain assumptions which we discuss in detail, it is possible to retrieve the high-frequency component of the Green's function between the two observation points. This in turn provides an estimate of the travel time between the two observation points.

The idea of using the cross correlation of noisy signals to retrieve information about travel times was first proposed in helioseismology and seismology [11, 21]. Physical derivations are discussed in [19, 6, 20, 24]. To summarize, these results are based on (1) the equipartition of mode energy, which means in particular that all spatial modes of the system, in a bounded domain, have uncorrelated energies, and (2) on ensemble averages with respect to modal fluctuations. In a homogeneous space without boundaries and with space-time white-noise sources, the Green's function can also be obtained from cross correlations, as is shown in [23]. An interesting application of this way of estimating Green's functions, and hence travel times, is for carrying out surface wave velocity estimation in seismology, as is done in [15] and in the references cited there.

In this paper we consider a bounded region for which the classical flow associated with the wave equation is ergodic, and we show that the high-frequency part of the Green's function can be retrieved from the cross-correlation of the noisy signals even when the spatial support of the noise sources is very small. The main ingredients for the derivation of these results are the following ones. First, the full wave dynamics is approximated by classical dynamics, and second, the ergodicity of the classical trajectories (rays) is used to prove convergence of the derivative of the cross correlation of the noisy signals to the Green's function. Semiclassical analysis was first used by Colin de Verdière to show how the cross correlation is related to the Green's function [8]. We follow here the same main steps, and carry out in detail the application of the ergodic theorem. This is combined with a precise estimate of convergence to the semiclassical approximation using Egoroff's theorem [12, 16].

Semiclassical analysis was introduced in order to prove the correspondence principle, which states that the quantum evolution of an observable can be approximated by its classical counterpart in the limit where Planck's constant \hbar is small. The mathematical formulation of this result is the Egorov theorem [12], whose proof for bounded time intervals is well established [16]. However, here we need a quantitative estimate of the long-time behavior of the semiclassical approximation. The time of validity of the semiclassical approximation is called the Ehrenfest time. It is of the order of $|\log \hbar|$ [26]. This qualitative estimate is analyzed mathematically in [10, 1, 5], where the semiclassical approximation is bounded by a term of the form $C_{\text{ego}} \hbar \exp(t/T_{\text{ego}})$ for times comparable

to the Ehrenfest time. The exponential in time growth of the error (with a rate given by the Egorov time T_{ego}) is also an important issue in quantum chaos [28]. This is because it is necessary to compare the long-time dynamical properties of the quantization of the classical flow with those of the quantum flow.

Semiclassical analysis can also be applied to high-frequency wave propagation to replace full wave dynamics by classical dynamics (geometric optics). In this setting, the small parameter is the ratio of the carrier wavelength to the typical propagation distance, or the diameter of the domain. The geometrical optics approximation is not valid beyond the Ehrenfest time. The Egorov time controls the exponential growth of the error between wave dynamics and geometric optics. Semiclassical analysis is used, for example, in the study of time reversal for waves in ergodic cavities in [2]. It is well-known that time reversal is a physical way to calculate cross correlation [13, 25, 22, 14], which explains its connection to this work. We do not use time-reversal ideas here. The purpose of our work is to show in detail that it is possible to apply the ergodic theorem to the classical flow while controlling the error in the semiclassical approximation.

We have pointed out that the approximation of full wave dynamics by geometric optics is dominated by a term that grows exponentially with time. This indicates that the application of the ergodic theorem to the ray dynamics does not give immediately the desired result, unless rates of convergence are known. However, it is not possible to obtain uniform estimate of the rate of convergence in the ergodic theorem, that is, an estimate that depends only on some L^p -norm of the averaged function (even the L^∞ -norm, see [17]). In this paper we apply carefully the ergodic theorem and show that a rate of convergence is needed only for the spatial power spectral density of the noise sources. This allows us to obtain an explicit expression of the cross correlation, which gives, in addition, resolution limits for travel time estimation.

This paper is self-contained and requires no detailed knowledge of semiclassical analysis. It is organized as follows. We formulate the problem in Section 2, where we present the ergodic hypothesis, the semiclassical Egorov theorem, and the observable operator that is the candidate for estimator of the Green's function. In Section 3 we show that the averaged operator is indeed close to the Green's function, considered as an operator, up to a remainder that is small when high-frequency or singular test functions are used. In Section 4 we prove that the operator is self-averaging, in the sense that its fluctuations go to zero as the averaging time goes to infinity.

2. Cross-Correlation of Recorded Signals

2.1. Green's Function Identification

We consider the solution u of the wave equation with attenuation,

$$\left(\frac{1}{T_a} + \partial_t\right)^2 u - \Delta u = n(t, x), \quad (1)$$

in a bounded open set $\Omega \subset \mathbb{R}^d$. For simplicity, the Dirichlet boundary condition $u = 0$ on $\partial\Omega$ is assumed. Here the operator Δ is defined by

$$\Delta = \nabla \cdot (c^2(x)\nabla), \quad (2)$$

which in a homogeneous medium $c(x) \equiv c_0$ is equal to c_0^2 times the usual Laplace operator. The local propagation velocity $c(x)$ is assumed to be bounded from below and above by two positive constants.

The term $n(t, x)$ models a random distribution of noisy sources. It is a zero-mean stationary (in time) Gaussian process with autocorrelation function

$$\langle n(s, x_s)n(t, y_s) \rangle = \delta(t - s)\theta\left(\frac{x_s + y_s}{2}, x_s - y_s\right), \quad (3)$$

where $\langle \cdot \rangle$ stands for the statistical average. The function $x_s \mapsto \theta(x_s, 0)$ models the spatial distribution of the sources. It is assumed to be smooth, bounded, and compactly supported in Ω . The function $z_s \mapsto \theta(x_s, z_s)$ is the local spatial autocorrelation function of the sources. Its Fourier transform with respect to the variable z_s is the local power spectral density $\hat{\theta}(x_s, \xi)$. It is assumed to be smooth, bounded, and integrable.

The question addressed in this paper is the identification of the time-dependent Green's function $G(t, x, y)$, that is, the fundamental solution of the wave equation in the absence of attenuation

$$\partial_t^2 G - \Delta G = \delta(t)\delta(x - y), \quad (4)$$

starting from $G(0, x) = \partial_t G(0, x) = 0$, where Δ is the operator defined by (2). In the case of an infinite homogeneous medium, $c(x) \equiv c_0$, $\Omega = \mathbb{R}^d$, $d = 3$, the Green's function is given by

$$G(t, x, y) = \frac{1}{4\pi c_0^2 |x - y|} \delta\left(\frac{|x - y|}{c_0} - t\right).$$

Clearly $G(t, x, y) = 0$ for $t < \tau_{x,y}$ where $\tau_{x,y} = |x - y|/c_0$ is the travel time from x to y . The support of the singular part of this Green's function, which is that of the Green's function itself in this case, allows us to identify the travel time $\tau_{x,y}$. This property of the singular part of the Green's function holds also in smoothly varying media and in bounded domains [16, Ch 24].

In a bounded inhomogeneous medium, which is the case we consider here, the Green's function can be written in terms of the eigenvalues ω_n^2 and orthonormal eigenfunctions ϕ_n of $-\Delta$, namely,

$$-\Delta\phi_n = \omega_n^2\phi_n \text{ in } \Omega, \quad \phi_n = 0 \text{ on } \partial\Omega.$$

The Green's function is the distribution

$$G(t, x, y) = \begin{cases} \sum_{n=1}^{\infty} \frac{\sin(\omega_n t)}{\omega_n} \phi_n(x)\phi_n(y) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases} \quad (5)$$

which, for $t > 0$, is the kernel of the operator

$$\frac{\sin \sqrt{-\Delta} t}{\sqrt{-\Delta}}.$$

Note that $-\Delta$ is self-adjoint, so that $G(t, x, y) = G(t, y, x)$. This reciprocity property plays a key role in the following. The Green's function contains information about the minimum travel time $\tau_{x,y}$ from x to y , since it is zero for $t < \tau_{x,y}$. This remark follows from the hyperbolicity of the wave equation (4). The minimum travel time is defined by

$$\tau_{x,y} = \inf \left\{ \int_0^1 \frac{|\phi'(t)| dt}{c(\phi(t))}, \phi \in \mathcal{C}^2([0, 1], \Omega), \phi(0) = x, \phi(1) = y \right\}, \quad (6)$$

where the curve(s) minimizing the action (6) is a ray that satisfies the Euler-Lagrange equations [16]. Of particular interest is here the fact that the high-frequency (singular) part of G also contains the relevant information about the travel times. A rather direct way to see this is by noting that the WKB (Wentzel-Kramers-Brillouin) approximation of the Green's function in the frequency domain has, at high frequency, the form

$$\hat{G}(\omega, x, y) \sim \sum_j a^{(j)}(x, y) e^{i\omega\tau_{x,y}^{(j)}}.$$

Here the sum is over travel times corresponding to rays that make the action (6) stationary. Therefore, if we want to estimate the velocity $c(x)$ of the medium, then it makes sense to look for only the high-frequency part of the Green's function, which identifies the travel time. If the travel time between enough points in the region is known, then the propagation velocity can be estimated by least squares methods [3]. This is the motivation for this paper. We shall show that the singular component of the Green's function can be estimated from the cross correlation function of the signals recorded at different points.

The integral representation of the solution of the wave equation with attenuation (1) is

$$u(t, x) = \int_{\Omega} \int_{-\infty}^t n(s, x_s) G(t - s, x, x_s) e^{-\frac{1}{T_a}(t-s)} ds dx_s. \quad (7)$$

The cross correlation of the recorded signals at x and y is defined by

$$C_T(\tau, x, y) = \frac{1}{T} \int_0^T u(t, x) u(t + \tau, y) dt. \quad (8)$$

In the limiting case of space-time white noise sources, that is, the case in which $\theta(x, y) = \delta(y)$, the relation between the cross correlation function and the Green's function is a direct one.

Proposition 2.1 *If $\theta(x, y) = \delta(y)$, then the average of C_T is an even function in τ that does not depend on T . Its τ -derivative is given by*

$$\partial_{\tau} \langle C_T(\tau, x, y) \rangle = -\frac{T_a}{4} \text{sgn}(\tau) G(|\tau|, x, y) e^{-\frac{|\tau|}{T_a}}. \quad (9)$$

The τ -derivative of the average of $C_T(\tau, x, y)$ is a symmetrized version of the Green's function $G(\tau, x, y)$, up to the damping factor $\exp(-|\tau|/T_a)$. Note that $\langle C_T \rangle$ is an even function in τ , and therefore its τ -derivative is an odd function.

Proof. The proof is an adaptation of the derivations given in the physics literature [19] and the mathematical one given in [8]. Using (7) we get the following integral representation for the average of the cross correlation function

$$\begin{aligned} \langle C_T(\tau, x, y) \rangle &= \frac{1}{T} \int_0^T \int_{-\infty}^t \int_{-\infty}^{t+\tau} \int_{\Omega^2} \langle n(s, x_s) n(s', x'_s) \rangle \\ &\quad \times G(t-s, x, x_s) G(t+\tau-s', y, x'_s) e^{-\frac{t-s}{T_a} - \frac{t+\tau-s'}{T_a}} dx_s dx'_s ds' ds dt. \end{aligned}$$

The process n is delta-correlated in time and space, so that

$$\langle C_T(\tau, x, y) \rangle = \frac{1}{T} \int_0^T \int_{-\infty}^{\min(t, t+\tau)} \int_{\Omega} G(t-s, x, x_s) G(t+\tau-s, y, x_s) e^{-2\frac{t-s}{T_a}} dx_s ds dt e^{-\frac{\tau}{T_a}}.$$

We introduce $\tau_+ = \max(\tau, 0)$ and $\tau_- = \min(\tau, 0)$ and we make the change of variable $s \mapsto u = t + \tau_- - s$

$$\langle C_T(\tau, x, y) \rangle = \frac{1}{T} \int_0^T \int_0^{\infty} \int_{\Omega} G(u - \tau_-, x, x_s) G(u + \tau_+, y, x_s) e^{-2\frac{u}{T_a}} dx_s du dt e^{-\frac{|\tau|}{T_a}}.$$

Here we have used the relations $\tau_+ + \tau_- = \tau$ and $\tau_+ - \tau_- = |\tau|$. This shows that $\langle C_T \rangle$ does not depend on T :

$$\langle C_T(\tau, x, y) \rangle = \int_0^{\infty} \int_{\Omega} G(u - \tau_-, x, x_s) G(u + \tau_+, y, x_s) e^{-2\frac{u}{T_a}} dx_s du e^{-\frac{|\tau|}{T_a}}.$$

We next substitute the expansion (5) of the Green's function in terms of the eigenvalues and eigenfunctions of $-\Delta$:

$$\begin{aligned} \langle C_T(\tau, x, y) \rangle &= \sum_{n, n'=1}^{\infty} \int_0^{\infty} \frac{\sin(\omega_n(u - \tau_-)) \sin(\omega'_n(u + \tau_+))}{\omega_n \omega'_n} e^{-2\frac{u}{T_a}} du \\ &\quad \times \int_{\Omega} \phi_n(x) \phi_n(x_s) \phi_{n'}(x_s) \phi_{n'}(y) dx_s e^{-\frac{|\tau|}{T_a}}. \end{aligned}$$

From the orthonormality property of the eigenfunctions

$$\int_{\Omega} \phi_n(x_s) \phi_{n'}(x_s) dx_s = \delta_{nn'}$$

and a direct computation we have

$$\int_0^{\infty} \frac{\sin(\omega_n(u - \tau_-)) \sin(\omega_n(u + \tau_+))}{\omega_n \omega_n} e^{-2\frac{u}{T_a}} du = \frac{T_a^2 \omega_n T_a \cos(\omega_n \tau) + T_a \sin(\omega_n |\tau|)}{4\omega_n (1 + \omega_n^2 T_a^2)}.$$

As a result,

$$\langle C_T(\tau, x, y) \rangle = \sum_{n=1}^{\infty} \frac{T_a^2 \omega_n T_a \cos(\omega_n \tau) + \sin(\omega_n |\tau|)}{4\omega_n (1 + \omega_n^2 T_a^2)} \phi_n(x) \phi_n(y) e^{-\frac{|\tau|}{T_a}},$$

which shows that $\langle C_T \rangle$ is an even function in τ . Taking the derivative with respect to τ gives

$$\partial_{\tau} \langle C_T(\tau, x, y) \rangle = -\frac{T_a}{4} \operatorname{sgn}(\tau) \sum_{n=1}^{\infty} \frac{\sin(\omega_n |\tau|)}{\omega_n} \phi_n(x) \phi_n(y) e^{-\frac{|\tau|}{T_a}},$$

which can be identified with (5) as (9). \square

This proposition shows that the interesting quantity for estimating travel times is the kernel K_T defined by

$$K_T(\tau, x, y) = -\frac{4}{T_a} \partial_\tau C_T(\tau, x, y) = -\frac{4}{T_a T} \int_0^T u(t, x) \partial_\tau u(t + \tau, y) dt. \quad (10)$$

We will show that, up to the damping factor $\exp(-\tau/T_a)$, this kernel is close to the Green's function $G(\tau, x, y)$. This is so even when the sources are correlated and have small spatial support, provided that some simple and explicit conditions are fulfilled. The first key ingredient for the derivation of this result is the ergodicity of the classical Hamiltonian flow. The second ingredient is the semiclassical Egorov theorem [12, 16], which allows us to control the difference between the classical dynamics of the rays and the waves dynamics, in the high-frequency regime. We present these two parts of the analysis in the next two subsections.

2.2. Ergodicity of the Hamiltonian Flow

We introduce the (classical) Hamiltonian flow which describes the propagation of rays

$$(x, \xi) \mapsto e^{th}(x, \xi) = (x_t(x, \xi), \xi_t(x, \xi)),$$

with the Hamiltonian $h(x, \xi) = \sqrt{(c^2(x)\xi, \xi)} = c(x)|\xi|$:

$$\begin{aligned} \frac{dx_t}{dt} &= c(x_t) \frac{\xi_t}{|\xi_t|}, & x_0(x, \xi) &= x, \\ \frac{d\xi_t}{dt} &= -\nabla c(x_t) |\xi_t|, & \xi_0(x, \xi) &= \xi. \end{aligned}$$

This Hamiltonian is well-defined as long as the ray remains in Ω . We then introduce the broken Hamiltonian flow, for which the rays are reflected on the boundary $\partial\Omega$ according to the Snell's law, that is, the angle of reflection is equal to the angle of incidence. The flow takes values in the cotangent space $T^*\Omega$, with the metric defined by the symbol $c(x)|\xi|$. For $q > 0$, the hypersurface

$$S_q^*(\Omega) = \{(x, \xi) \in T^*\Omega, c(x)|\xi| = q\}$$

can be seen as an energy surface with energy $q > 0$. These hypersurfaces are invariant with respect to the Hamiltonian flow:

$$e^{th} : S_q^*(\Omega) \rightarrow S_q^*(\Omega).$$

In fact, because of the homogeneity of the Hamiltonian in ξ , the Hamiltonian flow is determined by its restriction on the cotangent spherical bundle $S^*(\Omega) = S_1^*(\Omega)$: If $(x, \xi) \in S_q^*(\Omega)$, then $(x, \xi/q) \in S^*(\Omega)$ and

$$x_t(x, \xi) = x_t(x, \xi/q), \quad \xi_t(x, \xi) = q\xi_t(x, \xi/q). \quad (11)$$

The Liouville measure $d\mu_q$ on $S_q^*(\Omega)$ is characterized by the relation

$$\int_{T^*\Omega} f(x, \xi) dx d\xi = \int_0^\infty dq \int_{S_q^*(\Omega)} f(x, \xi) d\mu_q(x, \xi).$$

By homogeneity we have $d\mu_q(x, \xi) = q^{d-1} d\mu(x, \xi/q)$ and

$$\int_{T^*\Omega} f(x, \xi) dx d\xi = \int_0^\infty dq q^{d-1} \int_{S^*(\Omega)} f(x, q\xi) d\mu(x, \xi). \quad (12)$$

More explicitly, the Liouville measure $d\mu(m)$ on $S^*(\Omega)$ is

$$\int_{S^*(\Omega)} f(m) d\mu(m) = \int_\Omega dx \int_{S^{d-1}} d\sigma(\xi) c^{-d}(x) f(x, c(x)^{-1}\xi), \quad (13)$$

where $d\sigma(\xi)$ is the uniform measure on the (Euclidean) unit sphere S^{d-1} . In particular

$$\mu(S^*(\Omega)) = \int_{S^*(\Omega)} d\mu(m) = |S^{d-1}| \int_\Omega c(x)^{-d} dx.$$

Definition 2.1 *The Hamiltonian flow e^{th} is said to be (classically) ergodic if for any $f \in L^\infty(S^*(\Omega))$ and for $m = (x, \xi)$ in a subset of full measure of $S^*(\Omega)$,*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(e^{sh}m) ds = \bar{f}, \quad (14)$$

where the average of f is defined by

$$\bar{f} = \frac{1}{\mu(S^*(\Omega))} \int_{S^*(\Omega)} f(m) d\mu(m). \quad (15)$$

From now on we assume that the classical Hamiltonian flow is ergodic. The main result of the next section is a quantitative estimate of the difference between the averaged operator $\langle K_T \rangle$ and the Green's function. It depends on the convergence rate for (14). It is well-known that it is not possible to obtain an estimate of the rate of convergence in any L^p -norm [17]. However, as we shall see below, only the convergence rate for the function $\hat{\theta}(x, \xi)$ is required in this paper.

Remark. It is known that ergodicity for the classical Hamiltonian flow implies quantum ergodicity [7, 27, 28, 29], that is, ergodicity for operators of the form $e^{it\sqrt{-\Delta}} P e^{-it\sqrt{-\Delta}}$ where P is a zero-order pseudodifferential operator. However, this is not strong enough for what is needed here, and the error term cannot be estimated quantitatively. The purpose of this paper is to obtain such a quantitative result.

2.3. Semiclassical Egorov's Theorem

The function θ is determined by the two-point statistics of the source distribution. The Fourier transform $\hat{\theta}(x, \xi)$ of the function $z \mapsto \theta(x, z)$,

$$\hat{\theta}(x, \xi) = \int \theta(x, z) e^{-i\xi \cdot z} dz,$$

is the local power spectral density of the sources. The function θ defines the covariance operator $\Theta : L^2(\Omega) \rightarrow L^2(\Omega)$

$$\Theta\psi(x) = \int \theta\left(\frac{x+y}{2}, x-y\right)\psi(y)dy, \quad (16)$$

which is a zero-order pseudodifferential operator with symbol $\hat{\theta}(x, \xi)$

$$\Theta = \text{Op}(\hat{\theta}(x, \xi)).$$

Here we have used the Weyl quantization Op defined by

$$\text{Op}(\hat{\theta}(x, \xi))\psi(x) = \frac{1}{(2\pi)^d} \int \hat{\theta}\left(\frac{x+y}{2}, \xi\right) e^{i\xi \cdot (x-y)} \psi(y) dy d\xi. \quad (17)$$

The semiclassical Egorov theorem [12] (valid also for a domain Ω with a boundary [16]) states that the operators

$$e^{it\sqrt{-\Delta}}\Theta e^{-it\sqrt{-\Delta}} - \text{Op}\left[\hat{\theta}(e^{th}(x, \xi))\right]$$

is continuous from $H^{-1}(\Omega)$ to $L^2(\Omega)$ with a norm which grows in t with an exponential rate. This exponential growth is an important problem in quantum chaos because it compares the long-time dynamical properties of the quantization of the classical flow e^{th} with that of the quantum flow $e^{it\sqrt{-\Delta}}$, in the Heisenberg representation. We have the following result.

Lemma 2.1 *There exist $C_{\text{ego}}, T_{\text{ego}} > 0$ such that for any $\psi \in H^{-1}(\Omega)$*

$$\left\| e^{\pm it\sqrt{-\Delta}}\Theta e^{\mp it\sqrt{-\Delta}}\psi - \text{Op}\left[\hat{\theta}(e^{\pm th}(x, \xi))\right]\psi \right\|_{L^2(\Omega)} \leq C_{\text{ego}} e^{\frac{t}{T_{\text{ego}}}} \|\psi\|_{H^{-1}(\Omega)}. \quad (18)$$

Moreover, we have the following estimates for C_{ego} and T_{ego} :

$$C_{\text{ego}} \leq C \|c\|_{W^{[d/2]+2, \infty}(\Omega)}, \quad \frac{1}{T_{\text{ego}}} \leq C \|c\|_{W^{[d/2]+3, \infty}(\Omega)}, \quad (19)$$

where

$$\|c\|_{W^{m, \infty}(\Omega)} = \sup_{x \in \Omega, |\alpha| \leq m} |D_x^\alpha c(x)|, \quad (20)$$

with $D_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$ for $\alpha = (\alpha_i)_{i=1, \dots, d} \in \mathbb{N}^d$, and $|\alpha| = \sum_{i=1}^d \alpha_i$.

In (19) the constant C depends also on the function θ . The proof of Lemma 2.1 is given in the Appendix. We will see that the estimates (18) and (19) are direct consequences of explicit expressions for the remainder in the WKB expansion given in [16].

3. The Averaged Operator

We consider in this section the averaged operator $\langle K_T(\tau) \rangle$, where $K_T(\tau)$ is given by (10). The averaging is with respect to the distribution of the sources. Using the form (3) of the autocorrelation function of the random process n , we have for any $\tau > 0$

$$\begin{aligned} \langle K_T(\tau, x, y) \rangle &= -\frac{4}{T_a} \int_{\Omega^2} \int_0^\infty \left[\left(\partial_\tau - \frac{1}{T_a} \right) G(s + \tau, y, x_s) \right] G(s, x, y_s) \\ &\quad \times \theta \left(\frac{x_s + y_s}{2}, x_s - y_s \right) e^{-\frac{1}{T_a}(\tau+2s)} ds dx_s dy_s, \end{aligned}$$

which is independent of T . This is an integral representation of the field cross correlation in terms of the autocorrelation of the noise sources. By the reciprocity property $G(s, x, y_s) = G(s, y_s, x)$, we have that $\langle K_T(\tau, x, y) \rangle$ is the kernel of the operator

$$\begin{aligned} \langle K_T(\tau) \rangle &= -\frac{4}{T_a} \int_0^\infty e^{-\frac{1}{T_a}(\tau+2s)} \frac{\sin(\sqrt{-\Delta}s)}{\sqrt{-\Delta}} \Theta \cos(\sqrt{-\Delta}(s + \tau)) ds \\ &\quad - \frac{4}{T_a^2} \int_0^\infty e^{-\frac{1}{T_a}(\tau+2s)} \frac{\sin(\sqrt{-\Delta}s)}{\sqrt{-\Delta}} \Theta \frac{\sin(\sqrt{-\Delta}(s + \tau))}{\sqrt{-\Delta}} ds, \end{aligned}$$

where Θ is the covariance operator defined by (16).

3.1. Statement of the main results

In Proposition 2.1 we considered the case of space-time white-noise sources. We restate this proposition in the operator framework of this section.

Proposition 3.1 *If $\theta(x, y) = \delta(y)$, then $\langle K_T(\tau, x, y) \rangle$ is the kernel of the operator*

$$e^{-\frac{\tau}{T_a}} \frac{\sin \sqrt{-\Delta} \tau}{\sqrt{-\Delta}} \tag{21}$$

for any $\tau > 0$.

As noted in Proposition 2.1, this shows that if the noise sources are uniformly distributed in the domain Ω and delta-correlated in both space and time, then the averaged operator $\langle K_T(\tau) \rangle$ is equal to the symmetrized in time Green's function operator, up to multiplication by a damping factor. As noted below (9), the kernel of the averaged autocorrelation operator is even in τ , and therefore its τ derivative is odd. In (21) we show the operator only for $\tau > 0$. Propositions 2.1 and 3.1 do not require any ergodic properties of the Hamiltonian flow.

The more interesting and realistic case is the one in which the spatial distribution of the sources is neither stationary in space nor delta-correlated. The main result of this paper is that when the noise source distribution is characterized by the covariance function θ in (3), then the statement of the previous proposition is still true in the ergodic case, up to a smoothing operator that depends on θ and up to a remainder that is small for high frequencies.

Theorem 3.1 *If $d = 2$ or 3 , $c \in W^{4,\infty}(\Omega)$, $\hat{\theta}$ is smooth, bounded, and integrable, and $T_a < 2T_{\text{ego}}$, then $\langle K_T(\tau, x, y) \rangle$ is the kernel of the operator*

$$e^{-\frac{\tau}{T_a}} K_\theta \frac{\sin \sqrt{-\Delta} \tau}{\sqrt{-\Delta}} + R(\tau) \quad (22)$$

for any $\tau > 0$, where K_θ is a θ -dependent smoothing operator defined for any $\psi \in L^2(\Omega)$ by

$$K_\theta \psi(x) = \int \psi(x-z) \frac{1}{c(x-\frac{z}{2})^d} k_\theta \left(\frac{z}{c(x-\frac{z}{2})} \right) dz, \quad (23)$$

$$k_\theta(y) = \frac{\int_\Omega \theta_{\text{iso}}(z', c(z')y) dz'}{\int_\Omega c(z')^{-d} dz'}, \quad (24)$$

$$\hat{\theta}_{\text{iso}}(z', \xi) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} \hat{\theta}(z', |\xi|\eta) d\sigma(\eta). \quad (25)$$

The remainder $R(\tau)$ is small in the sense that there exists a constant C_R that depends only on θ and $\|c\|_{W^{4,\infty}(\Omega)}$ such that for any $\psi \in L^2(\Omega)$,

$$\|R(\tau)\psi\|_{H^1(\Omega)} \leq C_R e^{-\frac{\tau}{T_a}} \left[r_{\text{erg}}(T_a) + \frac{1}{1 - \frac{T_a}{2T_{\text{ego}}}} \frac{\|\psi\|_{H^{-1}(\Omega)}}{\|\psi\|_{L^2(\Omega)}} \right] \|\psi\|_{L^2(\Omega)}, \quad (26)$$

where $r_{\text{erg}}(t)$ goes to 0 as $t \rightarrow \infty$.

In the next two subsections we discuss the form of the smoothing operator K_θ and that of the remainder R in this theorem. The proof of the theorem is given in Subsection 3.4.

3.2. The Smoothing Operator

The symbol of the smoothing operator K_θ is $\hat{k}_\theta(c(x)\xi)$ so that

$$K_\theta = \text{Op} \left[\hat{k}_\theta(c(x)\xi) \right], \quad (27)$$

where \hat{k}_θ is the Fourier transform of k_θ . By the Weyl quantization we have

$$K_\theta \psi(x) = \frac{1}{(2\pi)^d} \int \hat{k}_\theta \left(c \left(\frac{x+y}{2} \right) \xi \right) e^{i\xi \cdot (x-y)} \psi(y) dy d\xi,$$

where

$$\hat{k}_\theta(c(x)\xi) = \frac{\int_\Omega \hat{\theta}_{\text{iso}} \left(z, \frac{c(x)}{c(z)} \xi \right) c(z)^{-d} dz}{\int_\Omega c(z)^{-d} dz}. \quad (28)$$

Note that the symbol of K_θ depends on ξ through its modulus only, which means that the smoothing operator K_θ is isotropic.

The function θ_{iso} in the smoothing operator K_θ is an "isotropized" transformation (25) of θ . If $\hat{\theta}(z, \xi)$ depends on ξ only through $|\xi|$, the modulus of ξ , then we simply have $\theta_{\text{iso}} = \theta$.

Let us examine two particular cases.

1. In the limit case, not covered by the theorem, in which the sources are spatially uncorrelated, $\theta(x, y) = \theta_s(x)\delta(y)$, the operator K_θ is a multiplication operator

$$K_\theta\psi(x) = \frac{\int \theta_s(z)c(z)^{-d}dz}{\int_\Omega c(z)^{-d}dz} \times \psi(x).$$

This means that the Green's function can be recovered exactly, up to a multiplicative constant. If in addition $\theta_s \equiv 1$, then we get

$$K_\theta\psi(x) = \psi(x),$$

which is in agreement with Proposition 3.1. Note, however, that the delta-correlated case $\theta(x, y) = \theta_s(x)\delta(y)$ is not addressed by Theorem 3.1, where the power spectral density $\hat{\theta}(x, \xi)$ is required to be integrable.

2. When c is a constant, then the smoothing operator K_θ is a convolution operator, that is, multiplication in the Fourier domain:

$$\begin{aligned} K_\theta\psi(x) |_{c=c_0} &= \int \psi(x-y)k_\theta(y)dy, \\ \hat{k}_\theta(\xi) |_{c=c_0} &= \frac{\int_\Omega \int_{S^{d-1}} \hat{\theta}(z', |\xi|\eta) d\sigma(\eta) dz'}{|S^{d-1}||\Omega|}. \end{aligned}$$

If in addition $\hat{\theta}(z, \xi)$ is independent of z and depends on ξ through its modulus only,

$$\hat{\theta}(z, \xi) = \hat{\theta}_c(|\xi|),$$

then the Fourier transform of the smoothing density is $\hat{k}_\theta(\xi) = \hat{\theta}_c(|\xi|)$. In this special case there are sources everywhere in the domain Ω , but they are spatially correlated. For example, if $\hat{\theta}_c(|\xi|) = \exp(-l_c^2|\xi|^2)$, then l_c can be identified as the correlation radius of the sources and the smoothing operator K_θ is simply a Gaussian convolution kernel with the effective radius l_c .

To summarize, the support of the density k_θ of the smoothing operator K_θ has an effective radius that is of the order of the correlation radius of the sources. This effective radius determines the accuracy of the cross correlation method for estimating the singular support of the Green's function, as we show in the next subsection.

3.3. The Remainder Operator

The remainder $R(\tau)$ is small if the two terms inside the square brackets on the right side of (26) are small. The function $r_{\text{erg}}(t)$ is determined by the rate of convergence of the ergodic theorem for the function $\hat{\theta}$ of the classical Hamiltonian flow. The second term in (26) is determined by the error in the semiclassical approximation. As a result, the remainder $R(\tau)$ is small if the following two conditions are fulfilled:

(1) $T_{\text{erg}} \ll T_a$, which means that convergence to the ergodic regime for the smooth function $\hat{\theta}$ of the classical flow occurs for times smaller than the attenuation time T_a . Here we have denoted by T_{erg} a characteristic decay time of the function $t \mapsto r_{\text{erg}}(t)$.

(2) $T_a < 2T_{\text{ego}}$ and the test function ψ is singular, in the sense that it contains mainly high frequencies. Under these conditions, the high-frequency wave dynamics is controlled by the classical flow for propagation times that are of the order of the attenuation time T_a .

If we restrict the set of test functions to the high-frequency set

$$B_\delta = \left\{ \psi \in L^2(\Omega), \|\psi\|_{L^2(\Omega)} = 1, \|\psi\|_{H^{-1}(\Omega)} \leq \delta \right\}, \quad (29)$$

where $\delta \ll 1$, then we have the estimate

$$\left\| \left(\langle K_T(\tau) \rangle - K_\theta e^{-\frac{\tau}{T_a}} \frac{\sin \sqrt{-\Delta} \tau}{\sqrt{-\Delta}} \right) \psi \right\|_{H^1(\Omega)} \leq C_R e^{-\frac{\tau}{T_a}} \left[r_{\text{erg}}(T_a) + \frac{\delta}{1 - \frac{T_a}{2T_{\text{ego}}}} \right], \quad (30)$$

where now the right-hand side is small. This shows that the high-frequency component of the averaged operator $\langle K_T(\tau) \rangle$ is close to $\exp(-\tau/T_a) K_\theta G(\tau)$.

To summarize, if $T_{\text{erg}} \ll T_a < 2T_{\text{ego}}$, then detecting the first peak of $\tau \mapsto \langle K_T(\tau, x, y) \rangle$ gives an estimate of the travel time from x to y . The accuracy of this estimate depends on the resolution of the smoothing operator, which, in turn, depends on the correlation radius of the sources.

3.4. Proof of Theorem 3.1

The kernel of the operator $\langle K_T(\tau, x, y) \rangle$ is

$$\begin{aligned} \langle K_T(\tau) \rangle &= K_1 + K_2, \\ K_1 &= -\frac{4}{T_a} \int_0^\infty e^{-\frac{1}{T_a}(\tau+2s)} \frac{\sin(\sqrt{-\Delta}s)}{\sqrt{-\Delta}} \Theta \cos(\sqrt{-\Delta}(s+\tau)) ds, \\ K_2 &= -\frac{4}{T_a^2} \int_0^\infty e^{-\frac{1}{T_a}(\tau+2s)} \frac{\sin(\sqrt{-\Delta}s)}{\sqrt{-\Delta}} \Theta \frac{\sin(\sqrt{-\Delta}(s+\tau))}{\sqrt{-\Delta}} ds. \end{aligned}$$

We first study K_1 and write it in the form

$$K_1 = e^{-\frac{\tau}{T_a}} K_\theta \frac{\sin \sqrt{-\Delta} \tau}{\sqrt{-\Delta}} + \frac{e^{-\frac{\tau}{T_a}}}{\sqrt{-\Delta}} (R_1 + R_2 + R_3), \quad (31)$$

with

$$\begin{aligned} R_1 &= \frac{i}{T_a} \int_0^\infty \left[\text{Op}(\hat{\theta}(e^{sh}(x, \xi))) - K_\theta \right] e^{-\frac{2s}{T_a}} ds e^{-i\tau\sqrt{-\Delta}} \\ &\quad - \frac{i}{T_a} \int_0^\infty \left[\text{Op}(\hat{\theta}(e^{-sh}(x, \xi))) - K_\theta \right] e^{-\frac{2s}{T_a}} ds e^{i\tau\sqrt{-\Delta}}, \\ R_2 &= \frac{i}{T_a} \int_0^\infty \left[e^{-is\sqrt{-\Delta}} \Theta e^{is\sqrt{-\Delta}} - \text{Op}(\hat{\theta}(e^{-sh}(x, \xi))) \right] e^{-\frac{2s}{T_a}} ds e^{-i\tau\sqrt{-\Delta}} \\ &\quad - \frac{i}{T_a} \int_0^\infty \left[e^{is\sqrt{-\Delta}} \Theta e^{-is\sqrt{-\Delta}} - \text{Op}(\hat{\theta}(e^{sh}(x, \xi))) \right] e^{-\frac{2s}{T_a}} ds e^{i\tau\sqrt{-\Delta}}, \\ R_3 &= \frac{i}{T_a} \int_0^\infty e^{-is\sqrt{-\Delta}} \Theta e^{-is\sqrt{-\Delta}} e^{-\frac{2s}{T_a}} ds e^{-i\tau\sqrt{-\Delta}} \\ &\quad - \frac{i}{T_a} \int_0^\infty e^{is\sqrt{-\Delta}} \Theta e^{is\sqrt{-\Delta}} e^{-\frac{2s}{T_a}} ds e^{i\tau\sqrt{-\Delta}}. \end{aligned}$$

The proof of Theorem 3.1 consists in studying the three operators R_1 , R_2 , and R_3 . The term R_1 can be estimated by using the ergodicity of the classical Hamiltonian flow (see Step 1). The term R_2 can be estimated by the Egorov theorem (see Step 2). The term R_3 can be bounded by using the fact that it is an off-resonant term (see Step 3).

Step 1. The main idea is to apply the ergodic theorem to the symbol of the operator R_1 . Let us define

$$R_{11} = \frac{1}{T_a} \int_0^\infty \left[\text{Op}(\hat{\theta}(e^{sh}(x, \xi))) - K_\theta \right] e^{-\frac{2s}{T_a}} ds.$$

The explicit form of this operator is

$$\begin{aligned} R_{11}\psi(x) &= \frac{1}{(2\pi)^d} \int e^{i(x-y)\cdot\xi} Q\left(\frac{x+y}{2}, \xi\right) \psi(y) dy d\xi - \frac{1}{2} K_\theta \psi(x) \\ &= \frac{1}{\pi^d} \int e^{2i(x-z)\cdot\xi} Q(z, \xi) \psi(2z-x) dz d\xi - \frac{1}{2} K_\theta \psi(x), \end{aligned} \quad (32)$$

where

$$Q(z, \xi) = \frac{1}{T_a} \int_0^\infty \hat{\theta}(e^{sh}(z, \xi)) e^{-\frac{2s}{T_a}} ds.$$

The expression (32) for the operator R_{11} involves an integral over $T^*\Omega$. We will rewrite it as an integral over energy surfaces and over the hypersurface $S^*(\Omega)$ using the identity (12), which states that the integral over $T^*\Omega$ of a function F can be written as

$$\int F(z, \xi) dz d\xi = \int_0^\infty dq q^{d-1} \int_{S^*(\Omega)} F(z, q\xi) d\mu(z, \xi).$$

Next we need to express $Q(z, q\xi)$ as a function of (z, ξ) . If $(x, \xi) \in S^*(\Omega)$, then we have

$$\begin{aligned} Q(x, q\xi) &= \frac{1}{T_a} \int_0^\infty \hat{\theta}(x_s(x, q\xi), \xi_s(x, q\xi)) e^{-\frac{2s}{T_a}} ds \\ &= \frac{1}{T_a} \int_0^\infty \hat{\theta}(x_s(x, \xi), q\xi_s(x, \xi)) e^{-\frac{2s}{T_a}} ds \\ &= \frac{1}{T_a} \int_0^\infty \hat{\theta}_q(e^{sh}(x, \xi)) e^{-\frac{2s}{T_a}} ds, \end{aligned}$$

where we have used (11) and we have defined

$$\hat{\theta}_q(x, \xi) = \hat{\theta}(x, q\xi).$$

Therefore

$$\begin{aligned} R_{11}\psi(x) &= \frac{1}{\pi^d} \int_0^\infty dq q^{d-1} \int_{S^*(\Omega)} \left[\frac{1}{T_a} \int_0^\infty \hat{\theta}_q(e^{sh}(z, \xi)) e^{-\frac{2s}{T_a}} ds \right] \psi_{q,x}(z, \xi) d\mu(z, \xi) \\ &\quad - \frac{1}{2} K_\theta \psi(x), \end{aligned} \quad (33)$$

where we have defined

$$\psi_{q,x}(z, \xi) = e^{2qi(x-z)\cdot\xi} \psi(2z-x).$$

We want now to identify $\frac{1}{2}K_\theta\psi(x)$ as the "average" of the first term of the right-hand side of (33). If we define

$$\bar{\theta}_q = \frac{1}{\mu(S^*(\Omega))} \int_{S^*(\Omega)} \hat{\theta}_q(m) d\mu(m),$$

then we have by (12)

$$\frac{1}{\pi^d} \int_0^\infty dq q^{d-1} \int_{S^*(\Omega)} \bar{\theta}_q \psi_{q,x}(z, \xi) d\mu(z, \xi) = \frac{1}{\pi^d} \int \bar{\theta}_{c(z)|\xi|} \psi(2z - x) e^{2i(x-z)\cdot\xi} dz d\xi. \quad (34)$$

We first compute the integral in ξ . The computation uses the formula (13) for the Liouville measure over $S^*(\Omega)$:

$$\begin{aligned} \int \bar{\theta}_{c(z)|\xi|} e^{2i(x-z)\cdot\xi} d\xi &= \frac{1}{\mu(S^*(\Omega))} \int d\xi \int_{S^*(\Omega)} \hat{\theta}(z', c(z)|\xi|\xi') d\mu(z', \xi') e^{2i(x-z)\cdot\xi} \\ &= \frac{1}{\mu(S^*(\Omega))} \int d\xi \int_\Omega dz' \int_{S^{d-1}} d\sigma(\xi') \hat{\theta}\left(z', \frac{c(z)}{c(z')}|\xi|\xi'\right) c(z')^{-d} e^{2i(x-z)\cdot\xi} \\ &= \frac{|S^{d-1}|}{\mu(S^*(\Omega))} \int d\xi \int_\Omega dz' \hat{\theta}_{\text{iso}}\left(z', \frac{c(z)}{c(z')}\xi\right) c(z')^{-d} e^{2i(x-z)\cdot\xi} \\ &= \frac{1}{c(z)^d \int_\Omega c(z')^{-d} dz'} \int d\xi \int_\Omega dz' \hat{\theta}_{\text{iso}}(z', \xi) e^{i\frac{2(x-z)}{c(z)}c(z')\cdot\xi} \\ &= \frac{(2\pi)^d}{c(z)^d} \frac{1}{\int_\Omega c(z')^{-d} dz'} \int_\Omega dz' \theta_{\text{iso}}\left(z', \frac{2(x-z)}{c(z)}c(z')\right) \\ &= \frac{(2\pi)^d}{c(z)^d} k_\theta\left(\frac{2(x-z)}{c(z)}\right), \end{aligned}$$

with k_θ defined by (24). Substituting into (34) we get

$$\begin{aligned} \frac{1}{\pi^d} \int_0^\infty dq q^{d-1} \int_{S^*(\Omega)} \bar{\theta}_q \psi_{q,x}(z, \xi) d\mu(z, \xi) &= \int_\Omega \frac{2^d}{c(z)^d} k_\theta\left(\frac{2(x-z)}{c(z)}\right) \psi(2z - x) dz \\ &= \int_\Omega \frac{1}{c(x - \frac{z}{2})^d} k_\theta\left(\frac{z}{c(x - \frac{z}{2})}\right) \psi(x - z) dz \\ &= K_\theta \psi(x). \end{aligned}$$

Substituting into (33) we obtain

$$R_{11}\psi(x) = \frac{1}{\pi^d} \int_0^\infty dq q^{d-1} \left\{ \int_{S^*(\Omega)} Q_q(z, \xi) \psi_{q,x}(z, \xi) d\mu(z, \xi) \right\}, \quad (35)$$

with

$$Q_q(z, \xi) = \frac{1}{T_a} \int_0^\infty \left[\hat{\theta}_q(e^{sh}(z, \xi)) - \bar{\theta}_q \right] e^{-\frac{2s}{T_a}} ds.$$

Note that $\bar{\theta}_q$ is the average of the function $\hat{\theta}_q(z, \xi)$ over $S^*(\Omega)$, so that the term in the curly brackets of (35) is of the form (A.5) in Lemma Appendix A.2. Therefore, applying the third item of Lemma Appendix A.2, we have for any $q \in (0, \infty)$

$$\left| \int_{S^*(\Omega)} Q_q(z, \xi) \psi_{q,x}(z, \xi) d\mu(z, \xi) \right| \leq Cr_{\text{erg}}(T_a, \hat{\theta}_q) \|\psi_{q,x}\|_{L^2(S^*(\Omega))},$$

where r_{erg} is defined by (A.4). Since the $L^2(S^*(\Omega))$ -norm of $\psi_{q,x}(z, \xi)$ is bounded by $C\|\psi\|_{L^2(\Omega)}$ uniformly in (x, q) , we get

$$\|R_{11}\psi\|_{L^2(\Omega)} \leq C\tilde{r}_{\text{erg}}(T_a)\|\psi\|_{L^2(\Omega)}, \quad (36)$$

with

$$\tilde{r}_{\text{erg}}(T_a) = \int_0^\infty dq q^{d-1} r_{\text{erg}}(T_a, \hat{\theta}_q).$$

By the second item of Lemma Appendix A.2, we know that the function $r_{\text{erg}}(t, \hat{\theta}_q)$ goes to 0 as $t \rightarrow \infty$ for any q . This function is also bounded by $\sup_{(x, \xi) \in S^*(\Omega)} |\hat{\theta}(x, q\xi)|$, which decays fast enough in q to ensure the convergence of the integral in q . The dominated convergence theorem then shows that the function $\tilde{r}_{\text{erg}}(t)$ goes to 0 as $t \rightarrow \infty$.

The same estimate as (36) holds for

$$R_{12} = \frac{1}{T_a} \int_0^\infty \left[\text{Op}(\hat{\theta}(e^{-sh}(x, \xi))) - K_\theta \right] e^{-\frac{2s}{T_a}} ds,$$

and consequently for $R_1 = iR_{11}e^{-i\sqrt{-\Delta}\tau} - iR_{12}e^{-i\sqrt{-\Delta}\tau}$, since $e^{\pm i\sqrt{-\Delta}\tau}$ is bounded in $L^2(\Omega)$.

Step 2. By Lemma 2.1 there exists $C_{\text{ego}}, T_{\text{ego}}$ such that, for any $\psi \in H^{-1}(\Omega)$,

$$\|R_2\psi\|_{L^2(\Omega)} \leq \frac{C_{\text{ego}}}{T_a} \int_0^\infty \exp\left(\frac{t}{T_{\text{ego}}} - \frac{2t}{T_a}\right) dt \|\psi\|_{H^{-1}(\Omega)} = \frac{C_{\text{ego}}}{2 - \frac{T_a}{T_{\text{ego}}}} \|\psi\|_{H^{-1}(\Omega)},$$

provided that $T_a < 2T_{\text{ego}}$.

Step 3. Let us define

$$R_{31} = \frac{i}{T_a} \int_0^\infty e^{-is\sqrt{-\Delta}} \Theta e^{-is\sqrt{-\Delta}} e^{-\frac{2s}{T_a}} ds.$$

We use the eigenvalues ω_n^2 and orthonormal eigenfunctions $\phi_n \in L^2(\Omega)$ of the operator $-\Delta$ with the Dirichlet boundary condition. For any $\psi, \tilde{\psi} \in L^2(\Omega)$,

$$\begin{aligned} (\tilde{\psi} R_{31} \psi) &= \sum_{n,p} (\tilde{\psi} \phi_n) (\phi_n \Theta \phi_p) (\phi_p \psi) \frac{i}{T_a} \int_0^\infty e^{-is(\omega_n + \omega_p)} e^{-\frac{2s}{T_a}} ds \\ &= i \sum_{n,p} \frac{(\tilde{\psi} \phi_n) (\phi_n \Theta \phi_p) (\phi_p \psi)}{2 + i(\omega_n + \omega_p) T_a}, \end{aligned}$$

and therefore

$$\|R_{31}\psi\|_{L^2(\Omega)}^2 \leq \sup_p \sum_n \frac{(\phi_n \Theta \phi_p)^2}{4 + (\omega_n + \omega_p)^2 T_a^2} \|\psi\|_{L^2(\Omega)}^2.$$

By the second item of Lemma Appendix A.1 in the Appendix, if $d = 2$ or 3 , then there exists a constant C_θ that depends only on θ such that, for any $\psi \in L^2$,

$$\|R_{31}\psi\|_{L^2(\Omega)}^2 \leq \frac{C_\theta^2 \|c\|_{L^\infty(\Omega)}^2}{T_a^2} \|\psi\|_{L^2(\Omega)}^2. \quad (37)$$

The same estimate as (37) holds for

$$R_{32} = \frac{i}{T_a} \int_0^\infty e^{is\sqrt{-\Delta}} \Theta e^{is\sqrt{-\Delta}} e^{-\frac{2s}{T_a}} ds,$$

and consequently for $R_3 = R_{31}e^{-i\tau\sqrt{-\Delta}} - R_{32}e^{i\tau\sqrt{-\Delta}}$. This establishes the existence of a constant C_θ that depends only on θ such that for any $\psi \in L^2(\Omega)$,

$$\|R_3\psi\|_{L^2(\Omega)} \leq \frac{C_\theta \|c\|_{L^\infty(\Omega)}}{T_a} \|\psi\|_{L^2(\Omega)},$$

and eventually

$$\|(R_1 + R_2 + R_3)\psi\|_{L^2(\Omega)} \leq C_R \left[r_{\text{erg}}(T_a) + \frac{1}{1 - \frac{T_a}{2T_{\text{ergo}}}} \frac{\|\psi\|_{H^{-1}(\Omega)}}{\|\psi\|_{L^2(\Omega)}} \right] \|\psi\|_{L^2(\Omega)},$$

with

$$r_{\text{erg}}(t) = \tilde{r}_{\text{erg}}(t) + \frac{C_\theta}{t}.$$

This depends only on θ and the ergodic Hamiltonian flow, and goes to 0 as $t \rightarrow \infty$. The expression (31) of K_1 and the fact that $(-\Delta)^{-1/2}$ is a bounded operator from $L^2(\Omega)$ to $H^1(\Omega)$ then gives

$$\|K_1\psi\|_{H^1(\Omega)} \leq C_R \left[r_{\text{erg}}(T_a) + \frac{1}{1 - \frac{T_a}{2T_{\text{ergo}}}} \frac{\|\psi\|_{H^{-1}(\Omega)}}{\|\psi\|_{L^2(\Omega)}} \right] \|\psi\|_{L^2(\Omega)}.$$

Finally, the analysis of the operator K_2 goes along the same lines as the one in step 3, and we find that its norm, as an operator from $L^2(\Omega)$ to $H^1(\Omega)$, is bounded by C_θ/T_a .

4. Statistical Stability

4.1. Statement of the Main Result

Theorem 3.1 shows that the operator K_T is close to the Green's function after averaging with respect to the distribution of the stationary noisy signals emitted by the sources. Moreover, the averaged operator $\langle K_T \rangle$ is independent of the averaging time T . In this section we show that the operator K_T is statistically stable. This means that the fluctuations of the operator K_T about its average are small when the averaging time T is large. We carry out a detailed analysis of the second moment of the operator K_T and we show that the norm of the fluctuations of the operator K_T has small standard deviation compared to the norm of the averaged operator when $T \gg T_a$.

Proposition 4.1 *As an operator from $L^2(\Omega)$ on $H^1(\Omega)$, the operator K_T is self-averaging. More precisely, there exists $C_s > 0$ such that, for any $\tau > 0$ and for any $\psi \in L^2(\Omega)$,*

$$\left\langle \|(K_T(\tau) - \langle K_T(\tau) \rangle) \psi\|_{H^1(\Omega)}^2 \right\rangle \leq C_s \frac{T_a}{T} \|\psi\|_{L^2(\Omega)}^2. \quad (38)$$

From this proposition and the Chebychev inequality we have that for any $M > 0$, $\psi \in L^2(\Omega)$,

$$\mathbb{P} \left(\left\| K_T(\tau)\psi - \langle K_T(\tau) \rangle \psi \right\|_{H^1(\Omega)} \geq M \sqrt{\frac{T_a}{T}} \|\psi\|_{L^2(\Omega)} \right) \leq \frac{C_s}{M^2}. \quad (39)$$

If we restrict the set of test functions to the high-frequency set B_δ defined by (29), that is, $B_\delta = \{\psi \in L^2(\Omega), \|\psi\|_{L^2(\Omega)} = 1, \|\psi\|_{H^{-1}(\Omega)} \leq \delta\}$, then we have for any $M > 0$, $\psi \in B_\delta$,

$$\begin{aligned} \mathbb{P} \left(\left\| K_T(\tau)\psi - K_\theta e^{-\frac{\tau}{T_a}} \frac{\sin \sqrt{-\Delta} \tau}{\sqrt{-\Delta}} \psi \right\|_{H^1(\Omega)} \right. \\ \left. \geq M \sqrt{\frac{T_a}{T}} + C_R e^{-\frac{\tau}{T_a}} \left[r_{\text{erg}}(T_a) + \frac{\delta}{1 - \frac{T_a}{2T_{\text{erg}}}} \right] \right) \leq \frac{C_s}{M^2}. \end{aligned} \quad (40)$$

This shows that K_T is the kernel of the smoothed Green's function if the test functions contain mainly high frequencies and if

$$T \gg T_a \gg T_{\text{erg}},$$

where T_{erg} is the characteristic decay time of the ergodic function rate r_{erg} .

As we will see in the proof below, the fluctuations of K_T do not decay to zero as $\tau \rightarrow \infty$. There is a term in the variance of the fluctuations that does not depend on τ . As a consequence, it is possible to reconstruct the Green's function $G(\tau, x, y)$ only for $\tau \leq \tau_c$, where

$$\tau_c \simeq T_a \ln(T/T_a).$$

4.2. Proof of Proposition 4.1

The key ingredient for the computation of the second moment of the operator K_T is the well-known result that high-order moments of Gaussian processes can be expressed in terms of sums and products of second-order moments. Let $x, y, x', y' \in \Omega$. Then,

$$\begin{aligned} \langle K_T(\tau, x, y) K_T(\tau, x', y') \rangle &= \frac{16}{T^2 T_a^2} \int_0^T dt \int_0^T dt' \int_\Omega dx_s \int_\Omega dx'_s \int_\Omega dy_s \int_\Omega dy'_s \\ &\times \int_0^\infty ds \int_{-\tau}^\infty ds' \int_0^\infty du \int_{-\tau}^\infty du' e^{-\frac{1}{T_a}(2\tau+s+s'+u+u')} G(s, x, x_s) \\ &\times \left[\left(\partial_\tau - \frac{1}{T_a} \right) G(s' + \tau, y, x'_s) \right] G(u, x', y_s) \left[\left(\partial_\tau - \frac{1}{T_a} \right) G(u' + \tau, y', y'_s) \right] \\ &\times \langle n(t-s, x_s) n(t-u, y_s) n(t'-s', x'_s) n(t'-u', y'_s) \rangle. \end{aligned}$$

The fourth-order moment of the Gaussian random process n is

$$\begin{aligned} \langle n(s, x_s) n(u, y_s) n(s', x'_s) n(u', y'_s) \rangle \\ = \delta(s-s') \delta(u-u') \theta\left(\frac{x_s+x'_s}{2}, x_s-x'_s\right) \theta\left(\frac{y_s+y'_s}{2}, y_s-y'_s\right) \\ + \delta(s-u) \delta(s'-u') \theta\left(\frac{x_s+y_s}{2}, x_s-y_s\right) \theta\left(\frac{x'_s+y'_s}{2}, x'_s-y'_s\right) \end{aligned}$$

$$+ \delta(s - u')\delta(s' - u)\theta\left(\frac{x_s + y'_s}{2}, x_s - y'_s\right)\theta\left(\frac{x'_s + y_s}{2}, x'_s - y_s\right).$$

Consequently, we have

$$\begin{aligned} & \frac{1}{T^2} \int_0^T \int_0^T \langle n(t - s, x_s)n(t - u, y_s)n(t' - s', x'_s)n(t' - u', y'_s) \rangle dt' dt \\ &= \delta(s - s')\delta(u - u')\theta\left(\frac{x_s + x'_s}{2}, x_s - x'_s\right)\theta\left(\frac{y_s + y'_s}{2}, y_s - y'_s\right) \\ &+ \frac{(T - |u - s|)_+}{T^2}\delta(u - u' + s' - s)\theta\left(\frac{x_s + y_s}{2}, x_s - y_s\right)\theta\left(\frac{x'_s + y'_s}{2}, x'_s - y'_s\right) \\ &+ \frac{(T - |u - s'|)_+}{T^2}\delta(u - u' + s - s')\theta\left(\frac{x_s + y'_s}{2}, x_s - y'_s\right)\theta\left(\frac{x'_s + y_s}{2}, x'_s - y_s\right). \end{aligned}$$

and the second moment of K_T can be decomposed into the sum of three contributions:

$$\begin{aligned} \langle K_T(\tau, x, y)K_T(\tau, x', y') \rangle &= \langle K_T(\tau, x, y)K_T(\tau, x', y') \rangle_{\text{I}} + \langle K_T(\tau, x, y)K_T(\tau, x', y') \rangle_{\text{II}} \\ &+ \langle K_T(\tau, x, y)K_T(\tau, x', y') \rangle_{\text{III}}, \end{aligned}$$

with

$$\begin{aligned} \langle K_T(\tau, x, y)K_T(\tau, x', y') \rangle_{\text{I}} &= \frac{16}{T_a^2} \int_{\Omega} dx_s \int_{\Omega} dx'_s \int_{\Omega} dy_s \int_{\Omega} dy'_s \int_0^{\infty} ds \int_{-\tau}^{\infty} ds' \\ &\times \int_0^{\infty} du \int_{-\tau}^{\infty} du' e^{-\frac{1}{T_a}(2\tau+s+s'+u+u')} \delta(s - s')\delta(u - u') \\ &\times \theta\left(\frac{x_s + x'_s}{2}, x_s - x'_s\right) G(s, x, x_s) \left[\left(\partial_{\tau} - \frac{1}{T_a} \right) G(s' + \tau, y, x'_s) \right] \\ &\times \theta\left(\frac{y_s + y'_s}{2}, y_s - y'_s\right) G(u, x', y_s) \left[\left(\partial_{\tau} - \frac{1}{T_a} \right) G(u' + \tau, y', y'_s) \right], \\ \langle K_T(\tau, x, y)K_T(\tau, x', y') \rangle_{\text{II}} &= \frac{16}{T_a^2} \int_{\Omega} dx_s \int_{\Omega} dx'_s \int_{\Omega} dy_s \int_{\Omega} dy'_s \int_0^{\infty} ds \int_{-\tau}^{\infty} ds' \\ &\times \int_0^{\infty} du \int_{-\tau}^{\infty} du' e^{-\frac{1}{T_a}(2\tau+s+s'+u+u')} \frac{(T - |u - s|)_+}{T^2} \delta(u - u' + s' - s) \\ &\times \theta\left(\frac{x_s + y_s}{2}, x_s - y_s\right) G(s, x, x_s) G(u, x', y_s) \\ &\times \theta\left(\frac{x'_s + y'_s}{2}, x'_s - y'_s\right) \left[\left(\partial_{\tau} - \frac{1}{T_a} \right) G(s' + \tau, y, x'_s) \right] \left[\left(\partial_{\tau} - \frac{1}{T_a} \right) G(u' + \tau, y', y'_s) \right], \\ \langle K_T(\tau, x, y)K_T(\tau, x', y') \rangle_{\text{III}} &= \frac{16}{T_a^2} \int_{\Omega} dx_s \int_{\Omega} dx'_s \int_{\Omega} dy_s \int_{\Omega} dy'_s \int_0^{\infty} ds \int_{-\tau}^{\infty} ds' \\ &\times \int_0^{\infty} du \int_{-\tau}^{\infty} du' e^{-\frac{1}{T_a}(2\tau+s+s'+u+u')} \frac{(T - |u - s'|)_+}{T^2} \delta(u - u' + s - s') \\ &\times G(s, x, x_s) \theta\left(\frac{x_s + y'_s}{2}, x_s - y'_s\right) \left[\left(\partial_{\tau} - \frac{1}{T_a} \right) G(u' + \tau, y', y'_s) \right] \\ &\times \left[\left(\partial_{\tau} - \frac{1}{T_a} \right) G(s' + \tau, y, x'_s) \right] \theta\left(\frac{x'_s + y_s}{2}, x'_s - y_s\right) G(u, x', y_s) \end{aligned}$$

Estimate of $\langle K_T(\tau, x, y)K_T(\tau, x', y') \rangle_I$. It is easy to check that

$$\langle K_T(\tau, x, y)K_T(\tau, x', y') \rangle_I = \langle K_T(\tau, x, y) \rangle \langle K_T(\tau, x', y') \rangle. \quad (41)$$

Estimate of $\langle K_T(\tau, x, y)K_T(\tau, x', y't) \rangle_{II}$. After integration in u' and change of variable $s'' = s' + \tau$,

$$\begin{aligned} \langle K_T(\tau, x, y)K_T(\tau, x', y') \rangle_{II} &= \frac{16}{T_a^2} \int_{\Omega} dx_s \int_{\Omega} dx'_s \int_{\Omega} dy_s \int_{\Omega} dy'_s \int_0^{\infty} ds'' \int_0^{\infty} du \int_0^{u+s''} ds \\ &\times e^{-\frac{2}{T_a}(s''+u)} \frac{(T - |u - s|)_+}{T^2} G(s, x, x_s) \theta\left(\frac{x_s + y_s}{2}, x_s - y_s\right) G(u, y_s, x') \\ &\times \left[\left(\partial_{s''} - \frac{1}{T_a} \right) G(s'', y, x'_s) \right] \theta\left(\frac{x'_s + y'_s}{2}, x'_s - y'_s\right) \left[\left(\partial_{s''} - \frac{1}{T_a} \right) G(u + s'' - s, y'_s, y') \right], \end{aligned}$$

which is independent of τ . If $\psi, \tilde{\psi} \in L^2(\Omega)$, then $\langle (\psi K_T \tilde{\psi})^2 \rangle_{II}$ is a combination of terms of the form

$$\begin{aligned} \left\langle \left(\psi K_T \tilde{\psi} \right)^2 \right\rangle_{II} &= \frac{16}{T_a^2} \int_0^{\infty} ds'' \int_0^{\infty} du \int_0^{u+s''} ds e^{-\frac{2}{T_a}(s''+u)} \frac{(T - |u - s|)_+}{T^2} \\ &\times \left(\tilde{\psi} \frac{e^{i\pm\sqrt{-\Delta}u}}{\sqrt{-\Delta}} \Theta \frac{e^{i\pm\sqrt{-\Delta}s}}{\sqrt{-\Delta}} \tilde{\psi} \right) \left(\psi e^{i\pm\sqrt{-\Delta}s''} \Theta e^{i\pm\sqrt{-\Delta}(u+s''-s)} \psi \right), \end{aligned}$$

Therefore we have

$$\begin{aligned} \left\langle \left(\psi K_T \tilde{\psi} \right)^2 \right\rangle_{II} &\leq \frac{16}{T_a^2 T} \int_0^{\infty} ds'' \int_0^{\infty} du \int_0^{u+s''} ds e^{-\frac{2}{T_a}(s''+u)} \|\tilde{\psi}\|_{H^{-1}(\Omega)}^2 \|\psi\|_{L^2(\Omega)}^2 \\ &\leq \frac{T_a}{T} \|\tilde{\psi}\|_{H^{-1}(\Omega)}^2 \|\psi\|_{L^2(\Omega)}^2. \end{aligned} \quad (42)$$

Estimate of $\langle K_T(\tau, x, y)K_T(\tau, x', y') \rangle_{III}$. After integration in u' ,

$$\begin{aligned} \langle K_T(\tau, x, y)K_T(\tau, x', y') \rangle_{III} &= \int_{\Omega} dx_s \int_{\Omega} dx'_s \int_{\Omega} dy_s \int_{\Omega} dy'_s \int_0^{\infty} ds \int_0^{\infty} du \int_{-\tau}^{u+s+\tau} ds' \\ &\times e^{-\frac{2}{T_a}(\tau+s+u)} \frac{(T - |u - s'|)_+}{T^2} \\ &\times G(s, x, x_s) \theta\left(\frac{x_s + y'_s}{2}, x_s - y'_s\right) \left[\left(\partial_{\tau} - \frac{1}{T_a} \right) G(u + s - s' + \tau, y'_s, y') \right] \\ &\times G(u, x', x'_s) \theta\left(\frac{x'_s + y_s}{2}, x'_s - y_s\right) \left[\left(\partial_{s''} - \frac{1}{T_a} \right) G(s' + \tau, y_s, y) \right]. \end{aligned}$$

If $\psi, \tilde{\psi} \in L^2(\Omega)$, then $\langle (\psi K_T \tilde{\psi})^2 \rangle_{III}$ is a sum of terms of the form

$$\begin{aligned} \left\langle \left(\psi K_T \tilde{\psi} \right)^2 \right\rangle_{III} &= \int_0^{\infty} ds \int_0^{\infty} du \int_{-\tau}^{u+s+\tau} ds' e^{-\frac{2}{T_a}(\tau+s+u)} \frac{(T - |u - s'|)_+}{T^2} \\ &\times \left(\tilde{\psi} \frac{e^{i\pm\sqrt{-\Delta}s}}{\sqrt{-\Delta}} \Theta \frac{e^{i\pm\sqrt{-\Delta}u}}{\sqrt{-\Delta}} \tilde{\psi} \right) \left(\psi e^{i\pm\sqrt{-\Delta}(u+s-s'+\tau)} \Theta e^{i\pm\sqrt{-\Delta}(s'+\tau)} \psi \right). \end{aligned}$$

Therefore we have

$$\left\langle \left(\psi K_T \tilde{\psi} \right)^2 \right\rangle_{III} \leq K \frac{T_a}{T} \left(1 + \frac{2\tau}{T_a} \right) e^{-\frac{2\tau}{T_a}} \|\tilde{\psi}\|_{H^{-1}(\Omega)}^2 \|\psi\|_{L^2(\Omega)}^2. \quad (43)$$

Summary. Collecting the results on the three contributions of the second moment of K_T , we find that, for any $\psi \in L^2(\Omega)$, $\tilde{\psi} \in H^{-1}(\Omega)$,

$$\begin{aligned} \left\langle \left[\left(\psi K_T \tilde{\psi} \right) - \left(\psi \langle K_T \rangle \tilde{\psi} \right) \right]^2 \right\rangle &= \left\langle \left(\psi K_T \tilde{\psi} \right)^2 \right\rangle_{\text{II}} + \left\langle \left(\psi \langle K_T \rangle \tilde{\psi} \right)^2 \right\rangle_{\text{III}} \\ &\leq K \frac{T_a}{T} \|\psi\|_{L^2(\Omega)}^2 \|\tilde{\psi}\|_{H^{-1}(\Omega)}^2, \end{aligned} \quad (44)$$

which gives the desired result.

5. Conclusion

We have introduced conditions under which the cross correlation function of the noisy signals recorded at two observation points contains enough information to identify the singular component of the the Green's function between these points. These conditions are as follows.

(1) The sources are assumed to emit stationary random signals that are Gaussian white-noise in time. Their spatial distribution is assumed to have a smooth covariance that is, in general, localized.

(2) The classical Hamiltonian flow is assumed to be ergodic. Quantitative estimates involve the rate of convergence of the ergodic theorem for the spatial power spectral density of the sources evaluated along the flow.

(3) The averaging time T , the attenuation time T_a , and the ergodic time T_{erg} are assumed to satisfy $T \gg T_a \gg T_{\text{erg}}$. These relations among the various time scales come from the Egorov theorem (Lemma 2.1), the ergodic theorem (Definition 2.1), as well as the statistical properties of Gaussian processes.

The analysis in this paper justifies the use of the cross correlation approach for travel time estimation in a bounded domain with spatially limited, noisy sources. It also quantifies the statistical stability of this estimation method as well as its accuracy, which is shown to depend on the spatial correlation of the sources. Extensions of these results to vector waves and to surface waves are now under consideration [8].

Acknowledgements

We thank Yves Colin de Verdière for useful and stimulating discussions. The work of G. Papanicolaou was supported by grants ONR N00014-02-1-0088, 02-SC-ARO-1067-MOD 1 and NSF DMS-0354674-001.

Appendix A. Technical Lemmas

In this appendix we establish the technical lemmas that are used in the proofs of Theorem 3.1 and Proposition 4.1.

The next lemma gives estimates for the eigenvalues and eigenfunctions of the operator $-\Delta$ that are used in the proof of Theorem 3.1 to control the norm of the remainder.

Lemma Appendix A.1 1. *There exists a constant C that depends only on θ such that*

$$|(\phi_p \Theta \phi_n)| \leq \frac{C \|c\|_{L^\infty(\Omega)}}{\max(\omega_n, \omega_p)}. \quad (\text{A.1})$$

2. *If $d = 2$ or $d = 3$, then there exists a constant C_θ that depends only on θ such that*

$$\sup_{p \geq 1} \sum_{n=1}^{\infty} \frac{(\phi_p \Theta \phi_n)^2}{(\omega_p + \omega_n)^2} \leq C_\theta^2 \|c\|_{L^\infty(\Omega)}^2. \quad (\text{A.2})$$

Proof. We have $-\Delta \phi_n = \omega_n^2 \phi_n$ so that

$$\begin{aligned} \omega_n^2 (\phi_p \Theta \phi_n) &= -(\phi_p \Theta \Delta \phi_n) = - \int \phi_p(x) \theta\left(\frac{x+y}{2}, x-y\right) \nabla_y \cdot c(y) \nabla_y \phi_n(y) dy dx \\ &= \int \phi_p(x) \left(\frac{1}{2} \nabla_1 - \nabla_2\right) \theta\left(\frac{x+y}{2}, x-y\right) \cdot c(y) \nabla_y \phi_n(y) dy dx, \end{aligned}$$

where we have used the fact that θ is compactly supported in Ω . Therefore, using the Cauchy-Schwarz inequality,

$$\begin{aligned} \omega_n^2 |(\phi_p \Theta \phi_n)| &\leq \left[\int \phi_p(x)^2 c(y) |\nabla_y \phi_n(y)|^2 dy dx \right]^{1/2} \\ &\quad \times \left[\int \left| \left(\frac{1}{2} \nabla_1 - \nabla_2\right) \theta\left(\frac{x+y}{2}, x-y\right) \right|^2 c(y) dy dx \right]^{1/2}. \end{aligned}$$

On the one hand, since $\|\phi_p\|_{L^2(\Omega)} = 1$ and $-\Delta \phi_n = \omega_n^2 \phi_n$,

$$\int \phi_p(x)^2 c(y) |\nabla_y \phi_n(y)|^2 dy dx = - \int \phi_n(y) \nabla_y \cdot c(y) \nabla_y \phi_n(y) dy = \omega_n^2.$$

On the other hand,

$$\int \left| \left(\frac{1}{2} \nabla_1 - \nabla_2\right) \theta\left(\frac{x+y}{2}, x-y\right) \right|^2 c(y) dy dx \leq \|c\|_{L^\infty(\Omega)} \int \left| \left(\frac{1}{2} \nabla_1 - \nabla_2\right) \theta(x, y) \right|^2 dx dy,$$

which gives the result of the first item.

The proof of the second item is based on (A.1) and on the Weyl law on counting eigenvalues $\omega_n \sim \bar{\omega} n^{1/d}$ [16, Ch. 24]. The parameter $\bar{\omega}^{-d}$ is proportional to the volume of $S^*(\Omega)$. By (A.1) we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(\phi_p \Theta \phi_n)^2}{(\omega_p + \omega_n)^2} &= \sum_{n=1}^p \frac{(\phi_p \Theta \phi_n)^2}{(\omega_p + \omega_n)^2} + \sum_{n=p+1}^{\infty} \frac{(\phi_p \Theta \phi_n)^2}{(\omega_p + \omega_n)^2} \\ &\leq C \|c\|_{L^\infty(\Omega)}^2 \left\{ \sum_{n=1}^p \frac{1}{\omega_p^2 (\omega_p + \omega_n)^2} + \sum_{n=p+1}^{\infty} \frac{1}{\omega_n^2 (\omega_p + \omega_n)^2} \right\} \\ &\sim p^{1-4/d} \int_0^1 \frac{1}{(x^{2/d} + 1)^2} dx + p^{1-4/d} \int_1^\infty \frac{1}{x^{2/d} (x^{2/d} + 1)^2} dx, \end{aligned}$$

which is finite and uniformly bounded in p if $d \leq 3$. \square

The next lemma uses the ergodicity of the classical flow to get the convergence of a special class of integrals used in the proof of Theorem 3.1.

Lemma Appendix A.2 *Assume that the Hamiltonian flow is ergodic in the sense of (14). Let $f \in L^\infty(S^*(\Omega))$.*

1. *For any $\varepsilon \in (0, 1)$, there exists $A_\varepsilon \subset S^*(\Omega)$ such that $|A_\varepsilon^c| \leq \varepsilon^2 |S^*(\Omega)|$ and*

$$\lim_{t \rightarrow \infty} \left\| \frac{1}{t} \int_0^t f(e^{sh}) ds - \bar{f} \right\|_{L^\infty(A_\varepsilon)} = 0. \quad (\text{A.3})$$

2. *The function*

$$r_{\text{erg}}(t, f) = \inf_{\varepsilon \in (0, 1)} \left\{ 2 \|f\|_{L^\infty(S^*(\Omega))} \varepsilon + \left\| \frac{1}{t\varepsilon} \int_0^{t\varepsilon} [f(e^{sh}) - \bar{f}] ds \right\|_{L^\infty(A_\varepsilon)} \right\} \quad (\text{A.4})$$

goes to zero as $t \rightarrow \infty$.

3. *If we denote $C = 3|S^*(\Omega)|^{1/2}$, then for any $\psi \in L^2(S^*(\Omega))$, we have*

$$\left| \int_{S^*(\Omega)} \frac{1}{T_a} \int_0^\infty [f(e^{sh}m) - \bar{f}] e^{-\frac{2s}{T_a}} ds \psi(m) d\mu(m) \right| \leq C r_{\text{erg}}(T_a, f) \|\psi\|_{L^2(S^*(\Omega))}. \quad (\text{A.5})$$

The function $r_{\text{erg}}(t, f)$ is determined by the ergodicity of the classical Hamiltonian flow for averaging times of order t . It gives the convergence rate of the time average of the function f to its mean value \bar{f} .

Proof. The first item of the lemma is a direct application of the classical Egorov theorem.

The second item consists in showing that $r_{\text{erg}}(t, f) \rightarrow 0$ as $t \rightarrow \infty$. Let $\varepsilon \in (0, 1)$ be fixed. By the first item, we have

$$\lim_{t \rightarrow \infty} \left\| \frac{1}{t\varepsilon} \int_0^{t\varepsilon} [f(e^{sh}) - \bar{f}] ds \right\|_{L^\infty(A_\varepsilon)} = 0.$$

As a consequence,

$$\limsup_{t \rightarrow \infty} r_{\text{erg}}(t, f) \leq 2 \|f\|_{L^\infty(S^*(\Omega))} \varepsilon.$$

Since this holds for any $\varepsilon \in (0, 1)$, we have the desired result.

We now turn our attention to the third item. We denote $F(s, m) = f(e^{sh}m) - \bar{f}$. For any $t \in (0, T_a)$,

$$\int_0^\infty F(s, m) e^{-\frac{2s}{T_a}} ds = \sum_{k=0}^\infty \int_{kt}^{(k+1)t} F(s, m) e^{-\frac{2s}{T_a}} ds = \sum_{k=0}^\infty e^{-\frac{2kt}{T_a}} \times \int_0^t F_k(s, m) e^{-\frac{2s}{T_a}} ds,$$

where $F_k(s, m) = F(s+kt, m) = F(s, e^{kth}m)$. We decompose the terms in the right-hand side as

$$\int_0^t F_k(s, m) e^{-\frac{2s}{T_a}} ds = \int_0^t F_k(s, m) ds + \int_0^t F_k(s, m) (e^{-\frac{2s}{T_a}} - 1) ds$$

and we integrate with respect to the test function ψ :

$$\begin{aligned} & \left| \int_{S^*(\Omega)} \frac{1}{T_a} \int_0^\infty F(s, m) e^{-\frac{2s}{T_a}} ds \psi(m) d\mu(m) \right| \\ & \leq \frac{t}{T_a} \sum_{k=0}^\infty e^{-\frac{2kt}{T_a}} \times \left[\left| \int_{S^*(\Omega)} \frac{1}{t} \int_0^t F_k(s, m) ds \psi(m) d\mu(m) \right| \right. \\ & \quad \left. + \left| \int_{S^*(\Omega)} \frac{1}{t} \int_0^t F_k(s, m) (e^{-\frac{2s}{T_a}} - 1) ds \psi(m) d\mu(m) \right| \right]. \end{aligned}$$

Let $\varepsilon \in (0, 1)$ and define A_ε as in the first item. We have on the one hand

$$\begin{aligned}
 & \left| \int_{S^*(\Omega)} \frac{1}{t} \int_0^t F_k(s, m) ds \psi(m) d\mu(m) \right| = \left| \int_{S^*(\Omega)} \frac{1}{t} \int_0^t F(s, m) ds \psi(e^{-kth} m) d\mu(m) \right| \\
 & \leq \left| \int_{A_\varepsilon} \frac{1}{t} \int_0^t F(s, m) ds \psi(e^{-kth} m) d\mu(m) \right| + \left| \int_{A_\varepsilon^c} \frac{1}{t} \int_0^t F(s, m) ds \psi(e^{-kth} m) d\mu(m) \right| \\
 & \leq \left\| \frac{1}{t} \int_0^t F(s, \cdot) ds \right\|_{L^\infty(A_\varepsilon)} \|\psi\|_{L^1(S^*(\Omega))} + \|f\|_{L^\infty(S^*(\Omega))} \|\mathbf{1}_{A_\varepsilon^c} \psi(e^{-kth})\|_{L^1(S^*(\Omega))} \\
 & \leq \left\| \frac{1}{t} \int_0^t F(s, \cdot) ds \right\|_{L^\infty(A_\varepsilon)} \|\psi\|_{L^2(S^*(\Omega))} |S^*(\Omega)|^{1/2} + \|f\|_{L^\infty(S^*(\Omega))} |A_\varepsilon^c|^{1/2} \|\psi\|_{L^2(S^*(\Omega))},
 \end{aligned}$$

where we have used the Cauchy-Schwarz inequality in the last step. On the other hand, since $|e^{-2y} - 1| \leq 2y$ for any $y \in (0, 1)$, we have

$$\begin{aligned}
 & \left| \int_{S^*(\Omega)} \frac{1}{t} \int_0^t F_k(s, m) (e^{-\frac{2s}{T_a}} - 1) ds \psi(m) d\mu(m) \right| \leq \frac{1}{t} \int_0^t \frac{2s}{T_a} ds \|f\|_{L^\infty(S^*(\Omega))} \|\psi\|_{L^1(S^*(\Omega))} \\
 & \leq \frac{t}{T_a} \|f\|_{L^\infty(S^*(\Omega))} \|\psi\|_{L^2(S^*(\Omega))} |S^*(\Omega)|^{1/2}.
 \end{aligned}$$

Finally, we have, for any $t \in (0, T_a)$,

$$\frac{t}{T_a} \sum_{k=0}^{\infty} e^{-\frac{2kt}{T_a}} = \frac{t}{T_a} \frac{1}{1 - e^{-\frac{2t}{T_a}}} \leq \sup_{y \in (0, 1)} \frac{y}{1 - e^{-2y}} \leq \frac{1}{1 - e^{-2}} \leq \frac{3}{2},$$

which yields

$$\begin{aligned}
 & \left| \int_{S^*(\Omega)} \frac{1}{T_a} \int_0^\infty [f(e^{sh} m) - \bar{f}] e^{-\frac{2s}{T_a}} ds \psi(m) d\mu(m) \right| \\
 & \leq \frac{3}{2} \left\{ \left(\frac{t}{T_a} + \varepsilon \right) \|f\|_{L^\infty(S^*(\Omega))} + \left\| \frac{1}{t} \int_0^t [f(e^{sh}) - \bar{f}] ds \right\|_{L^\infty(A_\varepsilon)} \right\} |S^*(\Omega)|^{1/2} \|\psi\|_{L^2(S^*(\Omega))},
 \end{aligned}$$

where we have used the fact that $|A_\varepsilon^c| \leq \varepsilon^2 |S^*(\Omega)|$. Since this holds for any $t \in (0, T_a)$, we can choose $t = T_a \varepsilon$, which gives

$$\begin{aligned}
 & \left| \int_{S^*(\Omega)} \frac{1}{T_a} \int_0^\infty [f(e^{sh} m) - \bar{f}] e^{-\frac{2s}{T_a}} ds \psi(m) d\mu(m) \right| \\
 & \leq \frac{3}{2} \left\{ 2\varepsilon \|f\|_{L^\infty(S^*(\Omega))} + \left\| \frac{1}{\varepsilon T_a} \int_0^{\varepsilon T_a} [f(e^{sh}) - \bar{f}] ds \right\|_{L^\infty(A_\varepsilon)} \right\} |S^*(\Omega)|^{1/2} \|\psi\|_{L^2(S^*(\Omega))}.
 \end{aligned}$$

Since this inequality true for any $\varepsilon \in (0, 1)$, we can take the infimum over such ε in the right-hand side, and this establishes the desired result. \square

Finally we prove the estimate (18) and Lemma 2.1, that allow us to control the error in the Egorov theorem.

Proof. In this proof we use arguments taken from [16]. The symbol $h(x, \xi) = c(x)|\xi|$ is homogeneous of order one. We introduce the transported zero-order symbol

$$p(x, \xi, t) = \hat{\theta}(e^{th}(x, \xi)),$$

solution of the Hamiltonian system

$$\partial_t p + \{h, p\} = 0, \quad p(x, \xi, 0) = \hat{\theta}(x, \xi). \quad (\text{A.6})$$

We only consider symbols p with support near rays which are transverse to the boundary. The interaction with rays tangent to the boundary or issuing from singular points of a piecewise C^∞ boundary can be handled following the method of [29] and we focus on the main points of the proof.

Taking derivatives and using Gronwall lemma we deduce from (A.6) the estimate

$$\sup_{|\alpha|+|\beta|\leq m} |D_{x,\xi}^{\alpha,\beta} p(x, \xi, t)| \leq C_m \sup_{|\alpha|+|\beta|\leq m} |D_{x,\xi}^{\alpha,\beta} \hat{\theta}(x, \xi)| \exp\left(Ct \sup_{|\alpha|\leq m+1} |D_x^\alpha c(x)|\right). \quad (\text{A.7})$$

Next we use estimates given by Hormander (cf. Theorem 18.1.8 page 71, volume III and estimate (7.6.10) of Theorem 7.6.5 page 209, volume I) to obtain

$$\|R_p(t)\|_{H^{-1}(\Omega)\rightarrow L^2(\Omega)} \leq C \sup_{|\alpha|\leq [d/2]+2} (|D_x^\alpha c(x)|, |D_x^\alpha p(x, \xi, t)|, |D_\xi^\alpha p(x, \xi, t)|) \quad (\text{A.8})$$

for the remainder $R_p(t) = i[\text{Op}(p(x, \xi, t)), \text{Op}(h(x, \xi))] - \text{Op}(\{p, h\})$ and

$$\begin{aligned} & \|[\sqrt{-\Delta} - \text{Op}(h(x, \xi)), \text{Op}(p(x, \xi, t))]\|_{H^{-1}(\Omega)\rightarrow L^2(\Omega)} \\ & \leq C \sup_{|\alpha|\leq [d/2]+2} (|D_x^\alpha c(x)|, |D_x^\alpha p(x, \xi, t)|, |D_\xi^\alpha p(x, \xi, t)|). \end{aligned} \quad (\text{A.9})$$

For the proof of (A.8) we compute with the formula (18.1.15) of [16] the symbols of the operators $\text{Op}(p(x, \xi, t))\text{Op}(h(x, \xi))$ and $\text{Op}(h(x, \xi))\text{Op}(p(x, \xi, t))$ up to the order 1 and uses for the remainder R_p the estimate (7.6.10) of [16].

For (A.9) we start with the relation:

$$\sqrt{-\Delta} = \sqrt{\text{Op}(c(x)^2|\xi|^2)} = \text{Op}(c(x)|\xi|) + \text{Op}(\tilde{R}_1), \quad (\text{A.10})$$

with \tilde{R}_1 a zero-order symbol with coefficients computed in term of $c(x)|\xi|$ and its first-order derivatives. Then

$$[\sqrt{-\Delta} - \text{Op}(h(x, \xi)), \text{Op}(p(x, \xi, t))] = [\text{Op}(\tilde{R}_1), \text{Op}(p(x, \xi, t))]$$

is an operator of degree -1 and its norm is estimated as above (with (18.1.15) and (7.6.10) of [16] leading to (A.9)) which eventually with the estimate (A.7) gives:

$$\begin{aligned} & \|R_p(t)\|_{H^{-1}(\Omega)\rightarrow L^2(\Omega)} + \|[\sqrt{-\Delta} - \text{Op}(h(x, \xi)), \text{Op}(p(x, \xi, t))]\|_{H^{-1}(\Omega)\rightarrow L^2(\Omega)} \\ & \leq C \sup_{|\alpha|\leq [d/2]+2} (|D_x^\alpha c(x)|) \exp\left(Ct \sup_{|\alpha|\leq [d/2]+3} |D_x^\alpha c(x)|\right). \end{aligned} \quad (\text{A.11})$$

Eventually for

$$E(t) = e^{-it\sqrt{-\Delta}}\Theta e^{it\sqrt{-\Delta}} - \text{Op}(p(x, \xi, t)) \quad (\text{A.12})$$

we proceed by linear superposition and Gronwall estimate: The operator

$$E_1(t) = e^{-it\sqrt{-\Delta}}\Theta e^{it\sqrt{-\Delta}}$$

is solution of the equation:

$$\frac{dE_1(t)}{dt} + i[\sqrt{-\Delta}, E_1(t)] = 0.$$

The operator $E_2(t) = \text{Op}(p(t, x, \xi))$ is by construction solution of the equation

$$\frac{dE_2(t)}{dt} + i[\text{Op}(h(x, \xi)), E_2(t)] = R_p(t),$$

or

$$\frac{dE_2(t)}{dt} + i[\sqrt{-\Delta}, E_2(t)] = R_{tot}(t),$$

where $R_{tot}(t) = R_p(t) + i[\sqrt{-\Delta} - \text{Op}(h(x, \xi)), \text{Op}(p(x, \xi, t))]$. Therefore for the “error” $E(t) = E_2(t) - E_1(t)$ we have the equation:

$$\frac{dE(t)}{dt} + i[\sqrt{-\Delta}, E(t)] = R_{tot}(t).$$

Since $E(0) = 0$, we obtain by the Duhamel formula:

$$E(t) = \int_0^t e^{i(t-s)\sqrt{-\Delta}} R_{tot}(s) e^{i(t-s)\sqrt{-\Delta}} ds.$$

This implies that the operator $E(t)$ is linear and continuous from $H^{-1}(\Omega)$ to $L^2(\Omega)$ with a norm bounded by

$$\|E(t)\|_{H^{-1}(\Omega) \rightarrow L^2(\Omega)} \leq C_s \int_0^t \|R_{tot}(s)\|_{H^{-1}(\Omega) \rightarrow L^2(\Omega)} ds. \quad (\text{A.13})$$

Using (A.11) we obtain

$$\begin{aligned} \|R_{tot}(t)\|_{H^{-1}(\Omega) \rightarrow L^2(\Omega)} &\leq \|R_p(t)\|_{H^{-1}(\Omega) \rightarrow L^2(\Omega)} \\ &\quad + \|[\sqrt{-\Delta} - \text{Op}(h(x, \xi)), \text{Op}(p(x, \xi, t))]\|_{H^{-1}(\Omega) \rightarrow L^2(\Omega)} \\ &\leq \sup_{|\alpha| \leq [d/2] + 2} (|D_x^\alpha c(x)|) \exp\left(Ct \sup_{|\alpha| \leq [d/2] + 3} |D_x^\alpha c(x)|\right), \end{aligned}$$

which gives, after substitution into (A.13),

$$\|E(t)\|_{H^{-1}(\Omega) \rightarrow L^2(\Omega)} \leq C \sup_{|\alpha| \leq [d/2] + 2} (|D_x^\alpha c(x)|) \exp\left(Ct \sup_{|\alpha| \leq [d/2] + 3} |D_x^\alpha c(x)|\right),$$

or in other words

$$\|e^{-it\sqrt{-\Delta}} \Theta e^{it\sqrt{-\Delta}} - \text{Op}(p(t, x, \xi))\|_{H^{-1}(\Omega) \rightarrow L^2(\Omega)} \leq C_{\text{ego}} \exp\left(\frac{t}{T_{\text{ego}}}\right), \quad (\text{A.14})$$

which is the desired result. \square

References

- [1] Bambusi D, Graffi S, and Paul T 1999 Long time semi-classical approximation of quantum flows: a proof of the Ehrenfest time *Asympt. Anal.* **21** 149-160.
- [2] Bardos C and Fink M 2002 Mathematical foundations of the time reversal mirror *Asympt. Anal.* **29** 157-182.
- [3] Berryman J 1990 Stable iterative reconstruction algorithm for nonlinear travel time tomography *Inverse Problems* **6** 21-42.

- [4] Borcea L, Papanicolaou G, and Tsogka C 2006 Adaptive interferometric imaging in clutter and optimal illumination *Inverse Problems* **22** 1405-1436.
- [5] Bouzouina A and Robert D 2002 Uniform semi-classical estimates for the propagation of quantum observables *Duke Math. J.* **111** 223-252.
- [6] Campillo M and Paul A 2003 Long range correlations in the seismic coda *Science* **29** 547-549.
- [7] Colin de Verdière Y 1985 Ergodicité et fonctions propres du Laplacien *Comm. Math. Phys.* **102** 497-502.
- [8] Colin de Verdière Y 2006 Mathematical models for passive imaging. I: general background *arXiv:math-ph/0610043v1*.
- [9] Colin de Verdière Y 2006 Mathematical models for passive imaging. II: Effective Hamiltonians associated to surface waves *arXiv:math-ph/0610044v1*.
- [10] Combescure M and Robert D 1999 Semi-classical spreading of quantum wave packets and applications near unstable fixed points of the classical flow *Asympt. Anal.* **14** 377-404.
- [11] Duvall Jr T L, Jefferies S M, Harvey J W, and Pomerantz M A 1993 Time-distance helioseismology *Nature* **362** 430-432.
- [12] Egorov Y V 1969 The canonical transformations of pseudodifferential operators *Uspehi Mat. Nauk* **25** 235-236.
- [13] Fink M, Cassereau D, Derode A, Prada C, Roux P, Tanter M, Thomas J L, and Wu F 2000 Time-reversed acoustics *Rep. Prog. Phys.* **63** 1933-1995.
- [14] Garnier J 2005 Imaging in randomly layered media by cross-correlating noisy signals *SIAM Multiscale Model. Simul.* **4** 610-640.
- [15] Gerstoft P, Sabra K G, Roux P, Kuperman W A, and Fehler M 2006 Green's function extraction and surface-wave tomography from microseisms in southern California *Geophysics* **71** SI23-SI31.
- [16] Hormander L 1980 *The analysis of linear partial differential operators* Vol. I and III (Berlin: Springer).
- [17] Krengel U 1985 *Ergodic theorems* (Berlin: Walter de Gruyter).
- [18] Larose E, Margerin L, Derode A, Van Tiggelen B, Campillo M, Shapiro N, Paul A, Stehly L, and Tanter M 2006 Correlation of random wave fields: an interdisciplinary review *Geophysics* **71** SI11-SI21.
- [19] Lobkis O I and Weaver R L 2001 On the emergence of the Green's function in the correlations of a diffuse field *J. Acoustic. Soc. Am.* **110** 3011-3017.
- [20] Malcolm A E, Scales J, and Van Tiggelen B A 2004 Extracting the Green function from diffuse, equipartitioned waves *Phys. Rev. E* **70** 015601.
- [21] Rickett J and Claerbout J 1999 Acoustic daylight imaging via spectral factorization: Helioseismology and reservoir monitoring *The Leading Edge* **18** 957-960.
- [22] Roux P and Fink M 2003, Green's function estimation using secondary sources in a shallow water environment *J. Acoust. Soc. Am.* **113** 1406-1416.
- [23] Roux P, Sabra K G, Kuperman W A, and Roux A 2005 Ambient noise cross correlation in free space: Theoretical approach *J. Acoust. Soc. Am.* **117** 79-84.
- [24] Snieder R 2004 Extracting the Green's function from the correlation of coda waves: A derivation based on stationary phase *Phys. Rev. E* **69** 046610.
- [25] Weaver R and Lobkis O I 2001 Ultrasonics without a source: Thermal fluctuation correlations at MHz frequencies *Phys. Rev. Lett.* **87** 134301.
- [26] Zaslavsky G M 1981 Stochasticity in quantum systems *Phys. Rep.* **80** 157-250.
- [27] Zelditch S 1987 Uniform distribution of eigenfunctions on compact hyperbolic surfaces *Duke Math. J.* **55** 919-941.
- [28] Zelditch S 1990 Quantum transition amplitudes for ergodic and for completely integrable systems *J. Funct. Anal.* **94** 415-436.
- [29] Zelditch S and Zworski M 1996 Ergodicity of eigenfunctions for ergodic billiards *Comm. Math. Phys.* **175** 673-682.