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High-frequency asymptotics for Maxwell's equations in anisotropic media Part I: Linear geometric and diffractive optics

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This paper is devoted to the derivation of the equations that govern the propagation of pulses in noncentrosymmetric crystals. The method is based upon high-frequency expansions techniques for Maxwell's equations. By suitable choices of the scalings we are able to derive two classical models: Geometric optics and diffractive optics (Schrödinger-type equations). In the so-called geometric regime we recover the standard results on the propagation of pulses in crystals (dispersion equation, polarization states, group velocity). In the diffractive regime we exhibit original results and give a closed-form expression for the diffraction operator which reads as an anisotropic operator. Given this expression we identify a critical configuration where the diffraction reduces to a one-dimensional second-order operator instead of the standard transverse Laplacian. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1354639]

I. INTRODUCTION

Many crystals and liquid crystals have optical properties which depend on the direction of propagation and the polarization of light. A precise understanding of light propagation in such anisotropic media is important for both theoretical and practical applications. Indeed this problem is theoretically interesting in that it exhibits many optical phenomena such as polarizations effects, optical rotation, and conical refraction.[1] It is also practically relevant since anisotropic media are essential components for many optical devices such as prism polarizers, birefringent filters, and Pockels cells.[2] Anisotropic media are also used for phase-matched frequency conversion.[3] The aim of this paper is to describe the effects of the anisotropy of the medium so as to derive evolution equations for the slowly varying envelopes of fields. Such results have been already obtained using more or less ad hoc methods (see Ref. 4 and references therein). In particular a modern method of solving optical problems is based on the integral formulation of the field equation and the determination of the Green function.[5] The method requires an explicit representation of the Green function which is obtained by the use of a Fourier transform. Then applying stationary phase method one gets the asymptotic form of the Green function. This method is efficient for linear media, but it is not well-adapted for addressing nonlinear problems since the use of Fourier transforms, and thus the derivation of an explicit form of the solution, are then prohibited.

We shall use a technique based on high-frequency expansions of the fields which has already been successfully applied to systems of linear, semilinear and quasilinear hyperbolic partial differential equations (see Ref. 6 and references therein). This technique is more robust than the Green function approach in the sense that the approximate solutions are not derived by an asymptotic analysis of an explicit solution, but through a direct asymptotic analysis of Maxwell's equations. This high-frequency asymptotics method can deal with boundary conditions and—more important—it can handle nonlinearities. Applying this technique to Maxwell's equations in anisotropic media allows us to get evolution equations for the field envelopes in both the scales

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corresponding to geometric optics and diffractive optics. We are then able to write equations which recover the classical results and also exhibit original results such as the explicit form of the anisotropic second-order operator which plays the role of the transverse Laplace operator in the standard Schrödinger equation.

The theoretical derivations of the equations and the physically relevant applications are quite long, but they can be divided into two parts. In this paper we restrict ourselves to linear propagation. Nonlinear propagation is addressed in the companion paper[7] that requires the results derived here below. The framework for high-frequency expansions of the solutions of Maxwell's equations follows from the appearance of the small parameter δ which has the order of magnitude of the carrier wavelength of light divided by the next smallest characteristic length present in the problem. If we consider the propagation along the z axis of a broadband and divergent pulse with carrier frequency ω, then there exists a wave number $k=k(\omega)$ such that the electric field $\mathcal{E}$ can be expanded as a series of slowly varying functions modulated by a rapid phase $\phi=kz-\omega t$

$$\mathcal{E}(t,x,y,z)=\frac{1}{2}\left(\sum_{j=0}^{\infty}\delta^j\mathbf{E}_j(\delta t,\delta x,\delta y,\delta z)e^{i(kz-\omega t)}+cc\right), \tag{1}$$

where cc is a shorthand for "complex conjugate." We are particularly interested in determining the leading profile $\mathbf{E}_0$, the so-called slowly varying envelope of the field. We shall derive the evolution equation for $\mathbf{E}_0$ by using the fact that it reads as the compatibility condition for the existence of the expansion (1). We shall see that for propagation length (z) of order δ^{-1} times the wavelength, which corresponds to the scales of the so-called geometric optics, evolution equations read as transport equations with constant velocity. Further, in the moving pulse-time frame (moving according to the velocity exhibited by the geometric transport equations), we can study the evolution of the field for propagation lengths of order δ^{-2} times the wavelength, which corresponds to the scales of the so-called diffractive optics. The evolution of the slowly varying envelope of the field is then governed by a Schrödinger-type equation.

The paper is organized as follows. First we describe the general configuration at hand in Sec. II. Sections III–V are devoted to an extensive study of the linear propagation of pulses in biaxial crystals. We finally apply these results to specific configurations and discuss some applications in the last sections of this paper.

II. FORMULATION AND SCALING

We aim at focusing on the derivation of the propagation equation, so we consider simple boundary conditions. We refer to Refs. 8 and 9 for extensive treatments of very general boundary conditions. In this paper the plane $\Sigma:=\{(x',y,z)\in\mathbb{R}^3,z=0\}$ is the boundary surface that separates the semi-infinite vacuum $\mathbb{R}^3_-:=\{(x,y,z)\in\mathbb{R}^3,z<0\}$ on the left and a biaxial crystal on the right $\mathbb{R}^3_+:=\{(x,y,z)\in\mathbb{R}^3,z>0\}$. We consider an incident beam incoming from the left whose propagation axis is perpendicular to the boundary surface Σ and is collinear to the z axis.

We assume absence of free charges or currents ($\mathbf{j}=0$, $\rho=0$), and that the crystal is nonmagnetic so that its magnetic permeability $\mu\equiv\mu_0$. Inside the domain $\mathbb{R}^3_+$, the electric field $\mathcal{E}$ and magnetic induction $\mathcal{B}$ obey the Maxwell equations

$$\partial_t\mathcal{B}+\mathbf{rot}\,\mathcal{E}=0,$$

$$\mu_0\partial_t\mathcal{D}-\mathbf{rot}\,\mathcal{B}=0,$$

where $\mathcal{D}$ is the electric induction which contains the physic interaction between light and matter and can be expressed in terms of $\mathcal{E}$. The magnetic field $\mathcal{H}$ is simply given by $\mathcal{B}=\mu_0\mathcal{H}$. By differentiating the second equation with respect to time t and substituting into the first one, the magnetic induction is eliminated so that we get the equation which governs the evolution of the electric field

$$\mathbf{rot}\,\mathbf{rot}\,\mathcal{E}=-\mu_0\partial_t^2\mathcal{D}. \tag{2}$$

Equation (2) is insufficient to determine the electric field and has to be supplemented by a constitutive equation showing how the field is related to the properties of the medium. Assuming that the wave intensity is small enough so that the response of the medium is linear, the electric induction reads

$$\mathcal{D}=\varepsilon_0\mathcal{E}+\mathcal{P} \tag{3}$$

$$\mathcal{P}=\varepsilon_0\chi^{(1)}*\mathcal{E}=\varepsilon_0\int_{-\infty}^{t}dt_1\chi^{(1)}(t-t_1)\mathcal{E}(t_1), \tag{4}$$

where $\mathcal{P}$ is the polarization of the medium. We assume that the electromagnetic wave is far enough from all absorption lines of the medium so that we can neglect absorption and the tensor $\chi^{(1)}$ is real and symmetric. ε_0 and μ_0 are, respectively, the dielectric constant and magnetic permeability of vacuum. These constant quantities are related to the light velocity c by the identity $\varepsilon_0\mu_0c^2=1$.

In order to deal with a well-posed problem we must state boundary conditions in time and space domains. The boundary condition at the boundary surface Σ is imposed by the continuity of the tangential components of the magnetic and electric fields. If we know the total field $\mathcal{S}_{\text{tot}}$ in vacuum just in the limit slab $z=0^-$, then the electric field $\mathcal{E}$ within the crystal in $z=0^+$ should satisfy $\mathcal{E}\times\mathbf{n}=\mathcal{S}_{\text{tot}}\times\mathbf{n}$, where $\mathbf{n}$ is the outgoing normal direction (0, 0, -1). Unfortunately we do not know *a priori* the total source, which divides into the sum of the incoming wave and the reflected wave. It is much more appropriate to consider as a boundary condition the incoming wave condition which is a well-adapted condition for almost normally incident pulses. The boundary conditions then read as the following equation over the interface Σ:

$$(\mathcal{E}-c\mathcal{B}\times\mathbf{n})\times\mathbf{n}=2\mathcal{S}\times\mathbf{n}, \tag{5}$$

where $\mathcal{S}$ is the source corresponding to the field of the incoming pulse given at the interface Σ by

$$\mathcal{S}(x,y,t)=\begin{pmatrix}\mathcal{S}_x(x,y,t)\\ \mathcal{S}_y(x,y,t)\\ 0\end{pmatrix}.$$

Last we assume that all unknown quantities are vanishing at time $t\leqslant 0$

$$\mathcal{E},\mathcal{B},\mathcal{D}(t=0)=0 \quad \text{in} \quad \mathbb{R}_+^3.$$

The source is assumed to be a modulation of a high-frequency signal whose carrier wavelength is λ_0, or the superposition of a finite number of such modes. From the characteristic spatial (resp. temporal) variations of the envelope of the source we can also define a length scale R_0 (resp. a time scale T_0). In order to make comparison we associate to the time scale T_0 the corresponding length scale $L_0:=cT_0$. Our study will take place in the framework where the dimensionless parameter $\delta=\min\{\lambda_0/R_0,\lambda_0/L_0\}$ is small. Note that this assumption prevents from addressing the cases of ultrashort pulses (whose duration is of the order of a few femtoseconds) and of ultrafocused beams (whose radius is of the order of a few micrometers). The most interesting case is then the configuration where R_0 and L_0 are of the same order: $\lambda_0\ll R_0\sim L_0$, since it is the case that contains all physical phenomena and the other configurations $\lambda_0\ll R_0\ll L_0$ and $\lambda_0\ll L_0\ll R_0$ can be deduced from the first one by straightforward approximations. Setting $\delta=\lambda_0/R_0$, $\tilde{x}=x/\lambda_0$, $\tilde{y}=y/\lambda_0$, $\tilde{z}=z/\lambda_0$, and $\tilde{t}=ct/\lambda_0$, the dimensionless Maxwell equation reads as:

$$\widetilde{\mathbf{rot}}\,\widetilde{\mathbf{rot}}\,\mathcal{E}=-\tilde{\mu}_0\partial_{\tilde{t}}^2\mathcal{D},$$

where $\tilde{\mu}_0 = \mu_0 c^2$. If we denote $\tilde{\varepsilon}_0 = \varepsilon_0$ and $\tilde{c} = 1$, then we still have the conservation relation $\tilde{\mu}_0 \tilde{\varepsilon}_0 \tilde{c}^2 = 1$. The source $\mathcal{S}$ has a high-frequency expansion of the form

$$\mathcal{S}(\tilde{x},\tilde{y},\tilde{t}) = \frac{1}{2} \sum_{\omega_f \in \Omega_{\mathcal{S}}} \begin{pmatrix} v_x^f(\delta\tilde{t},\delta\tilde{x},\delta\tilde{y}) \\ v_y^f(\delta\tilde{t},\delta\tilde{x},\delta\tilde{y}) \\ 0 \end{pmatrix} e^{-i\omega_f \tilde{t}} + cc. \tag{6}$$

$\Omega_{\mathcal{S}}$ is the collection (with finite cardinality) of the high carrier frequencies ω_f of the modes that the source contains. These carrier frequencies are now of order 1. $\mathbf{v}^f$ is the slowly varying envelope of the mode with carrier frequency ω_f, and the typical scale of the variations of the smooth function $(T,X,Y) \mapsto \mathbf{v}^f(T,X,Y)$ is of order 1. Note that a dimensionless propagation distance $\tilde{z}$ of the order of δ^{-1} corresponds to a physical distance of the order of R_0. Further a dimensionless propagation distance $\tilde{z}$ of the order of δ^{-2} corresponds to a physical distance of the order of R_0^2/λ_0 which is the well-known Rayleigh distance.

From now on we drop the tildes. We assume *a priori* that the electric field, the electric induction and the polarization can be expanded in a power series of the small parameter δ and in a series with respect to a set of rapid phases $k^f z - \omega_f t$:

$$\mathcal{E} = \frac{1}{2} \sum_{(\omega_f,k^f) \in H} (\mathbf{E}^f(\delta t,\delta x,\delta y,\delta z) e^{i(k^f z - \omega_f t)} + cc), \tag{7a}$$

$$\mathbf{E}^f(T,X,Y,Z) = \sum_{j=0}^{\infty} \delta^j \mathbf{E}_j^f(T,X,Y,Z), \tag{7b}$$

where $\mathbf{E}^f$ is the slowly varying envelope of the mode whose rapid phase is (ω_f,k^f). The functions $\mathbf{E}_j^f$ are smooth in all their arguments. H denotes the set of the rapid phases (ω_f,k^f) which are contained in the field $\mathcal{E}$. Finally note that the slow variables will be denoted throughout the paper by capital letters (say T), while the fast variables, or microscopic, will be represented by lower case letters (say t).

III. LINEAR POLARIZATION

A. Geometry

We introduce the geometric framework. We first define a reference frame (x, y, z) associated with the pulse where the carrier wave vector of the incoming pulse is collinear to the z axis. We then introduce a reference frame (1, 2, 3) associated with the optic axis of the crystal, where $\mathbf{e}_3$ is the main optic axis. The description of the carrier wave vector in the crystal reference frame is given in Fig. 1. θ stands for the angle between the wave vector $\mathbf{k}_0$ and the main optic axis. ϕ is the angle between the projection of the wave vector onto the plane $(\mathbf{e}_1,\mathbf{e}_2)$ and the axis collinear to $\mathbf{e}_1$. In such a configuration the transition matrix between the reference frames (x, y, z) and (1, 2, 3) is

$$U := \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

The matrix U is unitary and satisfies $U^{-1} = U^T$. Throughout the paper we use the notation M^T for the transpose of a matrix M. If $\mathbf{v}$ is a row vector (resp. line vector), $\mathbf{v}^T$ stands for the line vector (resp. row vector) whose coefficients are v_j.

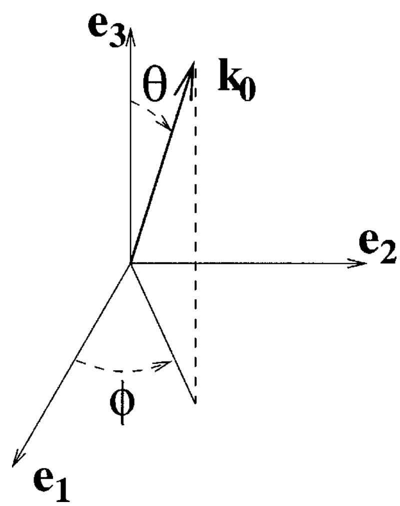

FIG. 1. Description of the wave vector in the crystallographic reference frame.

B. Linear susceptibility

The linear susceptibility is defined as the Fourier transform of the $\chi^{(1)}$ tensor[10]

$$\hat{\chi}^{(1)}(\omega) := \int_0^\infty dt_1 \chi^{(1)}(t_1) e^{i\omega t_1}.$$

Time integration starts from 0 to satisfy the causality property. In the crystallographic reference frame (1, 2, 3) the linear susceptibility is diagonal and reads:

$$\hat{\chi}_{123}^{(1)} = \begin{pmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix},$$

so that in the reference frame (x, y, z) the tensor $\hat{\chi}_{xyz}^{(1)}$ is $U^{-1}\hat{\chi}_{123}^{(1)}U$

$$\hat{\chi}_{xyz}^{(1)} = \begin{pmatrix} \chi_4 \cos^2\theta + \chi_3 \sin^2\theta & \frac{\sin(2\phi)}{2}(\chi_2-\chi_1)\cos\theta & \frac{\sin(2\theta)}{2}(\chi_4-\chi_3) \\ \frac{\sin(2\phi)}{2}(\chi_2-\chi_1)\cos\theta & \chi_1 \sin^2\phi + \chi_2 \cos^2\phi & \frac{\sin(2\phi)}{2}(\chi_2-\chi_1)\sin\theta \\ \frac{\sin(2\theta)}{2}(\chi_4-\chi_3) & \frac{\sin(2\phi)}{2}(\chi_2-\chi_1)\sin\theta & \chi_4 \sin^2\theta + \chi_3 \cos^2\theta \end{pmatrix}, \quad (8)$$

where $\chi_4 = \chi_1 \cos^2\phi + \chi_2 \sin^2\phi$. In the following χ is a shorthand for the matrix $\hat{\chi}_{xyz}^{(1)} + I_d$.

C. Expansion of the linear polarization

The linear induction is $\mathcal{D} = \varepsilon_0 \mathcal{E} + \mathcal{P}$. If $\mathcal{E} = \frac{1}{2}(\mathbf{E}(\delta t, \delta \mathbf{x}) e^{i(kz-\omega t)} + cc)$ and if we denote by $T = \delta t$ the slowly varying time variable, then the contribution of the linear induction to the Maxwell equation (2) can be expanded as

$$-\mu_0 \partial_t^2 \mathcal{D} = \tfrac{1}{2}(\mathbf{D}_0(\mathbf{E}) + \delta \mathbf{D}_1(\mathbf{E}) + \delta^2 \mathbf{D}_2(\mathbf{E}) + O(\delta^3)) e^{i(kz-\omega t)} + cc, \quad (9)$$

where the $\mathbf{D}_j(\mathbf{E})$ are linear functions of $\mathbf{E}$ given by

$$\mathbf{D}_0(\mathbf{E}) = \frac{\omega^2}{c^2}\chi\mathbf{E}, \quad \mathbf{D}_1(\mathbf{E}) = \frac{i}{c^2}(\omega^2\chi)'\,\partial_T\mathbf{E}, \quad \mathbf{D}^2(\mathbf{E}) = -\frac{1}{2c^2}(\omega^2\chi)''\,\partial_T^2\mathbf{E}, \tag{10}$$

and the primes indicate partial derivatives with respect to ω.

Proof: If $\mathcal{E} = \frac{1}{2}(\mathbf{E}(\delta t, \delta\mathbf{x})e^{i(kz-\omega t)} + cc)$, then by expanding the linear induction as powers of δ we get

$$\mathcal{D} = \tfrac{1}{2}(\mathbf{D}(\delta t, \delta\mathbf{x})e^{i(kz-\omega t)} + cc),$$

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \varepsilon_0\sum_{j=0}^{\infty}\frac{\delta^j(-1)^j}{j!}\left(\int_0^{\infty}dt'\,t'^j\,\chi^{(1)}(t')e^{i\omega t'}\right)\partial_T^j\mathbf{E}.$$

We introduce the derivatives of the linear susceptibility:

$$\hat{\chi}^{(1,j)}(\omega) = \frac{\partial^j\hat{\chi}^{(1)}}{\partial\omega^j}(\omega) = \int_0^{\infty}dt_1(it_1)^j\chi^{(1)}(t_1)e^{i\omega t_1},$$

so that the above expression reduces

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \varepsilon_0\sum_{j=0}^{\infty}\frac{\delta^j i^j}{j!}\hat{\chi}^{(1,j)}(\omega)\partial_T^j\mathbf{E}.$$

If only the coefficients of order smaller than δ^2 are retained, then the contribution of the linear polarization to the Maxwell equation reads

$$-\mu_0\partial_t^2\mathcal{D} = \frac{1}{2}\Bigg\{\frac{\omega^2}{c^2}(I_d + \hat{\chi}^{(1)})\mathbf{E} + i\delta\frac{\omega}{c^2}(\omega\hat{\chi}^{(1,1)} + 2\hat{\chi}^{(1)})\partial_T\mathbf{E}$$
$$-\delta^2\frac{1}{c^2}\left(\frac{1}{2}\omega^2\hat{\chi}^{(1,2)} + 2\omega\hat{\chi}^{(1,1)} + \hat{\chi}^{(1)}\right)\partial_T^2\mathbf{E} + O(\delta^3)\Bigg\}e^{i(kz-\omega t)} + cc,$$

which establishes the desired result since $\chi = I_d + \hat{\chi}^{(1)}$. □

Notice that an expansion of the field of the form $\mathcal{E} = \frac{1}{2}\mathbf{E}(\delta t, \delta\mathbf{x})e^{i(kz-\omega t)} + cc$ provides an expansion of the contribution of the linear induction of the form $\frac{1}{2}(\mathbf{D}_0(\mathbf{E}) + \delta\mathbf{D}_1(\mathbf{E}) + \delta^2\mathbf{D}_2(\mathbf{E}) + \cdots)e^{i(kz-\omega t)} + cc$. If one desires similar forms in both expansions, then it is sufficient to consider an expansion of the field of the form $\mathcal{E} = \frac{1}{2}(\mathbf{E}_0 + \delta\mathbf{E}_1 + \delta^2\mathbf{E}_2 + \cdots)(\delta t, \delta\mathbf{x})e^{i(kz-\omega t)} + cc$. This remark will appear determining in the establishing of a suitable ansatz for the solution of the Maxwell equation which is discussed in the next section.

IV. ELECTROMAGNETIC PROPAGATION IN ANISOTROPIC MEDIA

A. Principle of the high-frequency expansion

We aim at outlining the principle of the high-frequency expansion method. It can be applied if the source can be expanded as (6). Then we proceed to *a priori* expansions of the field inside the crystal of the form (7). In linear media the set of the frequencies ω which are contained in H is imposed by the source and is equal to $\Omega_{\mathcal{S}}$. In nonlinear media the generation of harmonics should be taken into account so that the set H could be much larger than in the linear case.

The establishing of the propagation equations for the slowly varying envelopes obeys the following scheme. The form (7) is substituted into Eq. (2): **rot rot** $\mathcal{E} = -\mu_0\partial_t^2\mathcal{D}$. We get the expansion with respect to δ by applying formulas (9) and (11) for the right-hand side of Eq. (2) (contribution of the linear induction) and the left-hand side of Eq. (2) (**rot rot** $\mathcal{E}$), respectively. Collecting the terms with similar orders in δ and the same rapid phases (ω_f, k^f), we get a family of equations. These equations can be decomposed into independent systems of equations parametrized by the rapid phases. The system corresponding to a rapid phase (ω_f, k^f) reads as a closed form system for the coefficients $\mathbf{E}_j^f$ of the series expansion of the envelope $\mathbf{E}^f$. This means that the envelopes of the different modes propagate independently. Note that in nonlinear media there may be coupling between the propagation equations of the modes. Considering the system for the mode with rapid phase (ω_f, k^f), we shall show on the one hand that a dispersion equation on (ω_f, k^f) appears as a compatibility condition for the existence of the high-frequency expansion (7a), and on the other hand that the form of the leading order term $\mathbf{E}_0^f$ is imposed by a compatibility condition for the existence of the series expansion (7b).

The form (7) is an ansatz, that is to say an *a priori* form of the solution which is valid in a given domain, here for $z \lesssim \delta^{-1}$. As an ansatz it satisfies basic properties. First it is compatible with the boundary conditions and the source. Second it is self-similar with respect to the operators that are encountered in the Maxwell equation. Indeed we have already established in Sec. III C that applying the operator corresponding to the right-hand side of the Maxwell equation to a an expansion of the kind (7) provides the same form. We are going to see in the next section devoted to the action of the **rot rot** operation that the expansion (7) is also self-similar with respect to this operator.

B. Expansion of the rot rot $\mathcal{E}$ term

If $\mathcal{E}$ is of the form $\mathcal{E} = \frac{1}{2}(\mathbf{E}(\delta t, \delta x, \delta y, \delta z) e^{i(kz-\omega t)} + cc)$, then **rot rot** $\mathcal{E}$ can be expanded as powers of δ. Denoting by $T = \delta t$, $X = \delta x$, $Y = \delta y$, and $Z = \delta z$ the slowly varying variables we find that

$$\textbf{rot rot}\,\mathcal{E} = \frac{1}{2}(\mathbf{R}_0(\mathbf{E}) + \delta \mathbf{R}_1(\mathbf{E}) + \delta^2 \mathbf{R}_2(\mathbf{E})) e^{i(kz-\omega t)} + cc, \tag{11}$$

where the mappings $\mathbf{R}_j(\mathbf{E})$ are given by

$$\mathbf{R}_0(\mathbf{E}) = \begin{pmatrix} k^2 E_x \\ k^2 E_y \\ 0 \end{pmatrix}, \tag{12a}$$

$$\mathbf{R}_1(\mathbf{E}) = \begin{pmatrix} ik\partial_X E_z - 2ik\partial_Z E_x \\ ik\partial_Y E_z - 2ik\partial_Z E_y \\ ik\partial_X E_x + ik\partial_Y E_y \end{pmatrix}, \tag{12b}$$

$$\mathbf{R}_2(\mathbf{E}) = \begin{pmatrix} -\partial_Y^2 E_x - \partial_Z^2 E_x + \partial_X \partial_Y E_y + \partial_X \partial_Z E_z \\ -\partial_X^2 E_y - \partial_Z^2 E_y + \partial_X \partial_Y E_x + \partial_Y \partial_Z E_z \\ -\partial_X^2 E_z - \partial_Y^2 E_z + \partial_X \partial_Z E_x + \partial_Y \partial_Z E_y \end{pmatrix}. \tag{12c}$$

Note that (11) actually holds true as an identity, and not only as an expansion. Furthermore $\mathbf{R}_2(\mathbf{E})$ is simply the standard ''**Rot Rot E**'' when the spatial derivatives are taken with respect to the slow variables (X, Y, Z).

C. Dispersion equation

Let us first assume that the input pulse has a single carrier frequency ω. By substituting the ansatz (7) into Eq. (2) and collecting the coefficients with power δ^0, we get by applying the identities (9) and (11) that $\mathbf{R}_0(\mathbf{E}_0) = \mathbf{D}_0(\mathbf{E}_0)$

$$k^2 J \mathbf{E}_0 = \omega^2 c^{-2} \chi \mathbf{E}_0, \tag{13}$$

where J is the matrix

$$J=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

The projection of Eq. (13) onto the z axis implies

$$E_{0z}=-\chi_{zz}^{-1}(\chi_{zx}E_{0x}+\chi_{zy}E_{0y}),$$

and substituting this identity into Eq. (13) we get that (E_{0x},E_{0y}) should fulfill

$$k^2\begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix}=\frac{\omega^2}{c^2}M\begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix},$$

where M is the 2×2 matrix

$$M=\begin{pmatrix} \chi_{xx}-\dfrac{\chi_{xz}\chi_{zx}}{\chi_{zz}} & \chi_{xy}-\dfrac{\chi_{xz}\chi_{zy}}{\chi_{zz}} \\ \chi_{yx}-\dfrac{\chi_{yz}\chi_{zx}}{\chi_{zz}} & \chi_{yy}-\dfrac{\chi_{yz}\chi_{zy}}{\chi_{zz}} \end{pmatrix}.$$

Note that, since χ is symmetric, M is also symmetric. The existence of a nonzero field $\mathbf{E}_0$ with carrier wave number k is equivalent to the Fresnel equation expressing the determinant of the system (13) equaling 0

$$\det(M-n^2I_d)=0. \tag{15}$$

The relationship $k=n\omega c^{-1}$ describes the normal waves that can propagate in the media with dielectric tensor χ. Eq. (15) also reads in terms of χ_1, χ_2, and χ_3 as

$$n^4-Sn^2+P=0,$$

$$S=\frac{(\chi_1\chi_2+\chi_3\chi_4)\sin^2\theta+\chi_3(\chi_1+\chi_2)\cos^2\theta}{\chi_3\cos^2\theta+\chi_4\sin^2\theta},$$

$$P=\frac{\chi_1\chi_2\chi_3}{\chi_3\cos^2\theta+\chi_4\sin^2\theta}.$$

The sum $n_a^2+n_b^2$ is equal to S and the product $n_a^2n_b^2$ is equal to P. Since S and P are positive it is easy to check that the Fresnel equation (15) has two positive solutions n_a and n_b. Consequently there exist two possible polarizations $\mathbf{s}_a$ and $\mathbf{s}_b$ so that $\mathbf{s}_a$ and $\mathbf{s}_b$ are unit vectors and $(n_a\omega c^{-1},\mathbf{s}_a)$ and $(n_b\omega c^{-1},\mathbf{s}_b)$ are solutions of Eq. (13). Furthermore the vectors (s_{ax},s_{ay}) and (s_{bx},s_{by}) are orthogonal. We define the dispersion relationship, the group velocity and the dispersion coefficient of the waves as follows:

$$k_m(\omega):=\frac{\omega n_m(\omega)}{c},\quad v_m(\omega):=\left(\frac{\partial k_m}{\partial\omega}\right)^{-1},\quad \sigma_m(\omega):=k_m\frac{\partial^2 k_m}{\partial\omega^2},\quad m=a,b. \tag{16}$$

Accordingly, if the polarization of the incoming pulse has components on both the x and y axis, then it should be decomposed into the sum of a type a and type b waves. We shall discuss this decomposition precisely in the next section. Further, if the incoming pulse is a superposition of several modes with different carrier frequencies, then for each carrier frequency ω_f the above results can be applied, so that each mode should be decomposed into the sum of a type *a* and type *b* waves.

Note that the occurrence of the case $n_a = n_b$ corresponds to very particular configurations. If the three indices χ_1, χ_2, and χ_3 are distinct (biaxial crystals), then the only cases for which $n_a = n_b$ are (assuming for instance $\chi_1 > \chi_2 > \chi_3$): $\phi = 0$ or π and $\theta = \pm \theta_c(\omega)$, where

$$\sin^2 \theta_c = \frac{\chi_1/\chi_2 - 1}{\chi_1/\chi_3 - 1}, \tag{17}$$

which defines the two optic axes of the biaxial crystal. If χ_3 is different from $\chi_1 = \chi_2$ (uniaxial crystals), then the only cases for which $n_a = n_b$ are $\theta = 0$ or π, and any ϕ. This defines the optic axis of the uniaxial crystal. Of course in isotropic medium $\chi_1 = \chi_2 = \chi_3$ one has $n_a = n_b$, but this trivial case will not be addressed in this paper.

Section V is devoted to an extensive study of the propagation of the pulse in anisotropic media in the general case when $n_a \neq n_b$. These results are then applied to the case of uniaxial crystals in Sec. VI. In Secs. VII–VIII we study the critical cases when $n_a = n_b$. Finally in Sec. IX we give some more results about the case when the crystal is biaxial but two of the crystal indices χ_j are close to each other.

D. Boundary condition

This condition reads as (5). We first eliminate the magnetic induction by differentiating with respect to time

$$(\partial_t \mathcal{E} + c\, \mathbf{rot}\, \mathcal{E} \times \mathbf{n}) \times \mathbf{n} = 2 \partial_t \mathcal{S} \times \mathbf{n}.$$

If we assume that the source $\mathcal{S}$ can be expanded as (6), and accordingly that the field $\mathcal{E}$ inside the crystal is of the form (7), then collecting the coefficients with power δ^0 and high carrier frequency ω_f establishes the continuity conditions which impose that the components parallel to the boundary surface of the input field $\mathcal{S}$ and of the field $\mathcal{E}$ should be equal, while there are no condition for the normal components. Consequently, the type m mode $(m = a,b)$ with carrier frequency ω_f should be at $z = 0^+$

$$\mathbf{E}^f_{0,m}(\delta t, \delta x, \delta y, z = 0^+) = c_{tr,m}(\omega_f) \mathbf{v}^f(\delta t, \delta x, \delta y),$$

where the transmission matrices $c_{tr,m}$ are

$$c_{tr,m}(\omega) = \frac{2}{1 + n_m(\omega)} \frac{1}{s_{mx}^2 + s_{my}^2} \begin{pmatrix} s_{mx}^2 & s_{mx}s_{my} & 0 \\ s_{mx}s_{my} & s_{my}^2 & 0 \\ s_{mx}s_{mz} & s_{my}s_{mz} & 0 \end{pmatrix}.$$

Note that, if the input field at frequency ω_f is linearly polarized along the $(s_{ax}, s_{ay}, 0)$-axis, then the field inside the crystal is purely type *a*. Moreover $c_{tr,m}(s_{mx}, s_{my}, 0)^T = (2/[1 + n_m(\omega)])\mathbf{s}_m$.

V. THE GEOMETRIC AND DIFFRACTIVE REGIMES IN THE GENERAL CASE

In this section we assume that the input pulse has only one carrier frequency ω and that the dispersion equation has two distinct solutions $n_a \neq n_b$. In such a configuration, the input pulse can be decomposed into a type *a* wave and a type *b* wave which propagate independently. We shall express our results in the case when the input pulse is linearly polarized according to the $(s_{ax}, s_{ay}, 0)$. The generalization to any polarization is then straightforward by application of the superposition principle. The case when $n_a = n_b$ require a specific study since the rapid phases of the type *a* and type *b* waves are equal, so their propagations may be coupled. This study will be carried out in Secs. VII and VIII.

A. Geometric optics

We assume in this section that the input pulse with carrier frequency ω is linearly polarized according to the $(s_{ax}(\omega),s_{ay}(\omega),0)$-axis

$$\mathcal{S}=\tfrac{1}{2}v(\delta x,\delta y,\delta t)(s_{ax},s_{ay},0)^T e^{-i\omega t}+cc. \tag{18}$$

Consequently we adopt the ansatz (7) with $H=\{(\omega,k_a(\omega))\}$

$$\mathcal{E}=\frac{1}{2}\left(\sum_{j=0}^{\infty}\delta^j\mathbf{E}_j(\delta t,\delta x,\delta y,\delta z)\right)e^{i(k_a(\omega)z-\omega t)}+cc. \tag{19}$$

We denote $T=\delta t$, $X=\delta x$, $Y=\delta y$, $Z=\delta z$, and k_a is a shorthand for $k_a(\omega)$.

Proposition 1. If the source $\mathcal{S}$ can be expanded as (18), then the leading order term $\mathbf{E}_0$ of the slowly varying envelope is linearly polarized along the $\mathbf{s}_a$-axis and satisfies the transport equation:

$$u_{ax}\partial_X E_0+u_{ay}\partial_Y E_o-\partial_Z E_0-v_a(\omega)^{-1}\partial_T E_0=0, \tag{20}$$

starting from $\mathbf{E}_0(T,X,Y,Z=0)=2/[1+n_a(\omega)]v(T,X,Y)$, *where* $\mathbf{u}_a=(s_{ax}s_{az}/(s_{ax}^2+s_{ay}^2),s_{ay}s_{az}/(s_{ax}^2+s_{ay}^2),-1)^T$. The solution of the above transport equation reads

$$E_0(T,X,Y,Z)=\frac{2}{1+n_a(\omega)}v(T-Z/v_a(\omega),X+u_{ax}Z,Y+u_{ay}Z).$$

It corresponds to the framework of the geometric optics, where the slowly varying envelope propagates without deformation with the group velocity. Note that the group velocity $v_a(\omega)$ and $v_b(\omega)$ are different. This phenomenon is called walk-off in the classical literature. It means that an input pulse will break into the sum of a type a and type b waves which propagate without interaction at different velocities. Furthermore the rays along which the waves propagate are not parallel. The Poynting vector of a type a wave is collinear to the vector $\mathbf{u}_a$ which means that the energy flow does not propagate along the direction of the carrier wave vector $\mathbf{k}_a$ which is collinear to the z axis. This is the so-called angular walk-off phenomenon. Finally note that $\mathbf{u}_a$ is orthogonal to the polarization vector $\mathbf{s}_a$: $\mathbf{u}_a.\mathbf{s}_a=0$, and that the three vectors $\mathbf{s}_a$, $\mathbf{u}_a$, and $\mathbf{k}_a$ lie in the same plane.

Proof: Let us substitute the ansatz (19) into Eq. (2) and collect the coefficients of each power of δ.

Order δ^0. The equations obtained at order δ^0 give the dispersion relationship and the fact that $\mathbf{E}_0=E_0\mathbf{s}_a$ (see Sec. IV C).

Order δ^1. The identity $\mathbf{R}_0(\mathbf{E}_1)+\mathbf{R}_1(\mathbf{E}_0)=\mathbf{D}_0(\mathbf{E}_1)+\mathbf{D}_1(\mathbf{E}_0)$ reads

$$\frac{\omega^2}{c^2}\chi\mathbf{E}_1+\frac{i}{c^2}(\omega^2\chi)'\partial_T\mathbf{E}_0=k_a^2J\mathbf{E}_1+ik_a\begin{pmatrix}\partial_X E_{0z}-2\partial_Z E_{0x}\\ \partial_Y E_{0z}-2\partial_Z E_{0y}\\ \partial_X E_{0x}+\partial_Y E_{0y}\end{pmatrix}, \tag{21}$$

where J is the matrix (14). We project this equation onto the vector $\mathbf{s}_a$. Since the matrix χ is symmetric, Eq. (13) implies

$$(\mathbf{s}_a^T)\chi-k_a^2(\mathbf{s}_a^T)J\equiv 0, \tag{22}$$

so that the terms in $\mathbf{E}_1$ cancel. Furthermore $\mathbf{E}_0=E_0\mathbf{s}_a$, so that it remains

$$\frac{i}{c^2}((\mathbf{s}_a^T)(\omega^2\chi)'\mathbf{s}_a)\partial_T E_0=2ik_a(s_{ax}s_{az}\partial_X E_0+s_{ay}s_{az}\partial_Y E_0-(s_{ax}^2+s_{ay}^2)\partial_Z E_0). \tag{23}$$

Besides, differentiating with respect to ω the equation $(\omega^2/c^2)\chi\mathbf{s}_a=k_a^2J\mathbf{s}_a$ yields

$$\frac{1}{c^2}(\omega^2\chi)\mathbf{s}_a+\frac{\omega^2}{c^2}\chi\mathbf{s}_a=k_a^2J\mathbf{s}_a+2k_ak_a'J\mathbf{s}_a. \tag{24}$$

Left-multiplying this equation by $\mathbf{s}_a$, the terms in $\mathbf{s}_a'$ cancel by (22), so it comes

$$c^{-2}((\mathbf{s}_a^T)(\omega^2\chi)'\mathbf{s}_a)=2k_ak_a'(s_{ax}^2+s_{ay}^2).$$

Substituting into (23) finally establishes Eq. (20). □

B. Diffractive optics

Eq. (20) describes the propagation of the envelope of the pulse for distances z of the order of δ^{-1}. The propagation is a pure transport without deformation. Consequently no evolution is noticeable in the moving reference frame $(\delta(t-z/v_a),\delta(x+u_{ax}z),\delta(y+u_{ay}z))$ when one looks at z of the order of δ^{-1}. We are now considering longer propagation distances z of the order of the Rayleigh distance δ^{-2} and we adopt the following ansatz:

$$\mathcal{E}=\frac{1}{2}\left(\sum_{j=0}^{\infty}\delta^j\mathbf{E}_j(\delta(t-z/v_a),\delta(x+u_{ax}z),\delta(y+u_{ay}z),\delta^2z)\right)e^{i(k_az-\omega t)}+cc. \tag{25}$$

We denote $T=\delta(t-z/v_a)$, $X=\delta(x+u_{ax}z)$, $Y=\delta(y+u_{ay}z)$ and the long scale variation of the envelope will be characterized by the variable $\zeta=\delta^2z$.

Proposition 2. If the source $\mathcal{S}$ can be expanded as (18), then the leading order term of the slowly varying envelope is linearly polarized along the $\mathbf{s}_a$-axis and satisfies in the moving frame the Schrödinger equation

$$\begin{aligned}2ik_a\partial_\zeta E_0+c_{a,xx}\partial_X^2E_0+c_{a,yy}\partial_Y^2E_0+2c_{a,xy}\partial_X\partial_YE_0-\sigma_a\partial_T^2E_0\\+2k_a(u_{ax}'\partial_T\partial_XE_0+u_{ay}'\partial_T\partial_YE_0)=0,\end{aligned} \tag{26}$$

starting from $E_0(T,X,Y,\zeta=0)=(2/[1+n_a(\omega)])v(T,X,Y)$, where the prime stands for the partial derivative with respect to ω. The diffraction coefficients are given by

$$c_{a,xx}(\omega)=\frac{n_a^2}{n_b^2-n_a^2}\left(\frac{\chi_{zy}}{\chi_{zz}}+2u_{ay}\right)^2+\frac{n_a^2s_{ax}^2}{\chi_{zz}(s_{ax}^2+s_{ay}^2)}+\frac{s_{ay}^2}{(s_{ax}^2+s_{ay}^2)^2},$$

$$c_{a,xy}(\omega)=-\frac{n_a^2}{n_b^2-n_a^2}\left(\frac{\chi_{zy}}{\chi_{zz}}+2u_{ay}\right)\left(\frac{\chi_{zx}}{\chi_{zz}+2u_{ax}}\right)+\frac{n_a^2s_{ax}s_{ay}}{\chi_{zz}(s_{ax}^2+s_{ay}^2)}-\frac{s_{ax}s_{ay}}{(s_{ax}^2+s_{ay}^2)^2},$$

$$c_{a,yy}(\omega)=\frac{n_a^2}{n_b^2-n_a^2}\left(\frac{\chi_{zx}}{\chi_{zz}}+2u_{ax}\right)^2+\frac{n_a^2s_{ay}^2}{\chi_{zz}(s_{ax}^2+s_{ay}^2)}+\frac{s_{ax}^2}{(s_{ax}^2+s_{ay}^2)^2}.$$

This proposition gives the equation which governs the propagation of the envelope of the field in the framework of the slowly varying envelope. We get here the result that the envelope in the moving framework satisfies a Schrödinger-type equation with an anisotropic diffraction operator, and that coupled time–space derivatives of the envelope are coming into the equation.

It thus appears that there exists a disagreement between the propagation equations which are proposed in standard references,[4] where the diffraction effect is represented as an isotropic Laplace operator with respect to the transverse coordinates, and our Eq. (26) where the diffraction effect reads as an anisotropic second order operator. We aim here at underlying the point where we feel the departure comes from. The direction of the polarization vector is assumed to be constant during propagation in previous derivations of the evolution equations. We have shown in this paper that this hypothesis holds true for the leading term $\mathbf{E}_0$, but it is wrong when considering the corrective term $\mathbf{E}_1$. We feel that the departure between the results essentially originates from the improper assumption about the constancy of the direction of the polarization vector.

The crossed space–time derivatives essentially originate from the fact that the Poynting vector of a monochromatic pulse is collinear to $\mathbf{u}_a(\omega)$ whose direction depends on the frequency ω. Accordingly the different frequencies of a broadband pulse (or equivalently a short pulse) do not propagate exactly in the same direction which involves this additive ''dispersion.''

PROOF: We substitute the ansatz (25) into Eq. (2) and we collect the coefficients with the same power of δ. In the expressions (12b) and (12c) of $\mathbf{R}_1$ and $\mathbf{R}_2$ we must take care to replace $\partial_Z(.)$ by $-v_a^{-1}\partial_T(.)+u_{ax}\partial_X(.)+u_{ay}\partial_Y(.)$ and to take into account the new slow variable ζ.

Order δ^1. Once rewritten in terms of the new variables, the transport equation (20) becomes trivial. Furthermore, since

$$((\omega^2\chi)'\mathbf{s}_a)_z=(\omega^2\chi\mathbf{s}_a)'_z-(\omega^2\chi\mathbf{s}'_a)_z=0-\omega^2(\chi_{zx}s'_{ax}+\chi_{zy}s'_{ay}+\chi_{zz}s'_{az}),$$

the projection of Eq. (21) onto the z axis establishes that

$$E_{1z}=-\chi_{zz}^{-1}(\chi_{zx}E_{1x}+\chi_{zy}E_{1y})+\frac{ik_ac^2}{\omega^2\chi_{zz}}(\partial_XE_{0x}+\partial_YE_{0y})+i\left(\frac{\chi_{zx}}{\chi_{zz}}s'_{ax}+\frac{\chi_{zy}}{\chi_{zz}}s'_{ay}+s'_{az}\right)\partial_TE_0. \tag{27}$$

Substituting into the projections of Eq. (21) onto the axes x and y

$$\left(\frac{\omega^2}{c^2}M-k_a^2I_d\right)\begin{pmatrix}E_{1x}\\E_{1y}\end{pmatrix}=ik_a\begin{pmatrix}-s_{ay}\\s_{ax}\end{pmatrix}\left(\left(-\frac{\chi_{yz}}{\chi_{zz}}-2u_{ay}\right)\partial_XE_0+\left(\frac{\chi_{xz}}{\chi_{zz}}+2u_{ax}\right)\partial_YE_0\right)$$
$$+i\left(\left(\frac{\omega^2}{c^2}M-k_a^2I_d\right)\begin{pmatrix}s'_{ax}\\s'_{ay}\end{pmatrix}-2k_ak'_a\begin{pmatrix}s_{ax}\\s_{ay}\end{pmatrix}\right)\partial_TE_0. \tag{28}$$

Since the vectors (s_{ax},s_{ay}) and $(-s_{ay},s_{ax})$ are orthogonal, there exist two scalars A_a and B_a such that

$$\begin{pmatrix}E_{1x}\\E_{1y}\end{pmatrix}=A_a\begin{pmatrix}s_{ax}\\s_{ay}\end{pmatrix}+B_a\begin{pmatrix}-s_{ay}\\s_{ax}\end{pmatrix}, \tag{29}$$

and two scalars C_a and D_a such that

$$\left(\frac{\omega^2}{c^2}M-k_a^2I_d\right)\begin{pmatrix}-s_{ay}\\s_{ax}\end{pmatrix}=C_a\begin{pmatrix}s_{ax}\\s_{ay}\end{pmatrix}+D_a\begin{pmatrix}-s_{ay}\\s_{ax}\end{pmatrix}.$$

Since $\binom{s_{ax}}{s_{ay}}$ is an eigenvector of $\omega^2c^{-2}M$ with eigenvalue k_a^2, left-multiplying this equation by $\binom{s_{ax}}{s_{ay}}$ yields: $0=C_a(s_{ax}^2+s_{ay}^2)$ so $C_a=0$. Thus $\binom{-s_{ay}}{s_{ax}}$ is an eigenvector of $\omega^2c^{-2}M$ with eigenvalue $D_a+k_a^2$, and by definition of the eigenvalues this proves that $D_a=k_b^2-k_a^2$. Eq. (28) now reads

$$B_a=i\frac{s_{ax}s'_{ay}-s_{ay}s'_{ax}}{s_{ax}^2+s_{ay}^2}\partial_TE_0+\frac{ik_a}{k_b^2-k_a^2}\left(\left(-\frac{\chi_{yz}}{\chi_{zz}}-2u_{ay}\right)\partial_XE_0+\left(\frac{\chi_{xz}}{\chi_{zz}}+2u_{ax}\right)\partial_YE_0\right). \tag{30}$$

Order δ^2. Collecting the coefficients of order δ^2 we get:

$$\mathbf{R}_0(\mathbf{E}_2)+\mathbf{R}_1(\mathbf{E}_1)+\mathbf{R}_2(\mathbf{E}_0)=\mathbf{D}_0(\mathbf{E}_2)+\mathbf{D}_1(\mathbf{E}_1)+\mathbf{D}_2(\mathbf{E}_0).$$

Projecting onto the vector $\mathbf{s}_a$ the terms in $\mathbf{E}_2$ cancel and it remains

$$\mathbf{s}_a.\mathbf{R}_1(\mathbf{E}_1)+\mathbf{s}_a.\mathbf{R}_2(\mathbf{E}_0)=\mathbf{s}_a.\mathbf{D}_1(\mathbf{E}_1)+\mathbf{s}_a.\mathbf{D}_2(\mathbf{E}_0). \tag{31}$$

We now compute the four terms of this identity.

(i) Computation of $\mathbf{S}_a.\mathbf{D}_1(\mathbf{E}_1)$.
By definition:

$$\mathbf{S}_a.\mathbf{D}_1(\mathbf{E}_1)=ic^{-2}(\mathbf{s}_a^T)(\omega^2\chi)'\partial_T\mathbf{E}_1.$$

From (22) this expression simplifies

$$\mathbf{s}_a.\mathbf{D}_1(\mathbf{E}_1)=2ik_ak_a'\partial_T(s_{ax}E_{1x}+s_{ay}E_{1y})-i(\mathbf{s}_a'^T)(\omega^2c^{-2}\chi-k_a^2J)\partial_T\mathbf{E}_1. \tag{32}$$

Differentiating Eq. (21) with respect to time and multiplying by $\mathbf{s}_a'^T$ establishes

$$(\mathbf{s}_a'^T)(\omega^2c^{-2}\chi-k_a^2J)\partial_T\mathbf{E}_1$$
$$=i(\mathbf{s}_a'^T)(\omega^2c^{-2}\chi-k_a^2J)\mathbf{s}_a'\partial_T^2E_0+ik_a(s_{ax}^2+s_{ay}^2)(u_{ax}'\partial_T\partial_XE_0+u_{ay}'\partial_T\partial_YE_0).$$

On the one hand, differentiating Eq. (24) with respect to ω and multiplying by $\mathbf{s}_a^T$

$$c^{-2}(\mathbf{s}_a^T)(\omega^2\chi)''\mathbf{s}_a+2c^{-2}(\mathbf{s}_a^T)(\omega^2\chi)'\mathbf{s}_a'$$
$$=4k_ak_a'(s_{ax}s_{ax}'+s_{ay}s_{ay}')+2((k_a')^2+k_ak_a'')(s_{ax}^2+s_{ay}^2).$$

On the other hand, multiplying Eq. (24) by $\mathbf{s}_a'^T$

$$c^{-2}(\mathbf{s}_a'^T)(\omega^2\chi)'\mathbf{s}_a+\omega^2c^{-2}(\mathbf{s}_a'^T)\chi\mathbf{s}_a'=k_a^2((s_{ax}')^2+(s_{ay}')^2)+2k_ak_a'(s_{ax}'s_{ax}+s_{ay}'s_{ay}).$$

Multiplying by 2 the last identity and subtracting the last two identities establish:

$$c^{-2}(\mathbf{s}_a^T)(\omega^2\chi)''\mathbf{s}_a=2(\mathbf{s}_a'^T)(\omega^2c^{-2}\chi-k_a^2J)\mathbf{s}_a'+2((k_a')^2+k_ak_a'')(s_{ax}^2+s_{ay}^2).$$

Finally, substituting into (32)

$$\mathbf{s}_a.\mathbf{D}_1(\mathbf{E}_1)=2ik_ak_a'\partial_T(s_{ax}E_{1x}+s_{ay}E_{1y})+k_a(s_{ax}^2+s_{ay}^2)(u_{ax}'\partial_T\partial_XE_0+u_{ay}'\partial_T\partial_YE_0)$$
$$+(-((k_a')^2+k_ak_a'')(s_{ax}^2+s_{ay}^2)+2^{-1}c^{-2}(\mathbf{s}_a^T)(\omega^2\chi)''\mathbf{s}_a)\partial_T^2E_0. \tag{33}$$

(ii) Computation of $\mathbf{s}_a.\mathbf{D}_2(\mathbf{E}_0)$.
By definition and using the fact that $\mathbf{E}_0$ reads $\mathbf{s}_aE_0$

$$\mathbf{s}_a.\mathbf{D}_2(\mathbf{E}_0)=-2^{-1}c^{-2}(\mathbf{s}_a^T)(\omega^2\chi)''\mathbf{s}_a\partial_T^2E_0. \tag{34}$$

(iii) Computation of $\mathbf{s}_a.\mathbf{R}_2(\mathbf{E}_0)$.
Computing $\mathbf{s}_a.\mathbf{R}_2(\mathbf{E}_0)$ is easy

$$\mathbf{s}_a.\mathbf{R}_2(\mathbf{E}_0)=\frac{-s_{ay}^2}{s_{ax}^2+s_{ay}^2}\partial_X^2E_0+\frac{2s_{ax}s_{ay}}{s_{ax}^2+s_{ay}^2}\partial_X\partial_YE_0+\frac{-s_{ax}^2}{s_{ax}^2+s_{ay}^2}\partial_Y^2E_0$$
$$-2ik_a(s_{ax}^2+s_{ay}^2)\partial_\zeta E_0-\frac{s_{ax}^2+s_{ay}^2}{v_a^2}\partial_T^2E_0. \tag{35}$$

(iv) Computation of $\mathbf{s}_a.\mathbf{R}_1(\mathbf{E}_1)$.
We first compute $\mathbf{s}_a.\mathbf{R}_1(\mathbf{E}_1)$ by taking into account (27)

$$\mathbf{s}_a.\mathbf{R}_1(\mathbf{E}_1)=ik_a\left[\left(-\frac{\chi_{zy}}{\chi_{zz}}-2u_{ay}\right)\partial_X+\left(\frac{\chi_{zx}}{\chi_{zz}}+2u_{ax}\right)\partial_Y\right](s_{ax}E_{1y}-s_{ay}E_{1x})$$
$$-\frac{k_a^2c^2}{\omega^2\chi_{zz}}(s_{ax}^2\partial_X^2E_0+2s_{ax}s_{ay}\partial_X\partial_YE_0+s_{ay}^2\partial_Y^2E_0)+2i\frac{k_a}{v_a}\partial_T(s_{ax}E_{1x}+s_{ay}E_{1y})$$
$$-k_a\left(\frac{\chi_{zx}}{\chi_{zz}}s'_{ax}+\frac{\chi_{zy}}{\chi_{zz}}s'_{ay}+s'_{az}\right)(s_{ax}\partial_T\partial_XE_0+s_{ay}\partial_T\partial_YE_0).$$

Using the representation (29) and the identity (30)

$$\mathbf{s}_a.\mathbf{R}_1(\mathbf{E}_1)=-\left[\frac{n_a^2(s_{ax}^2+s_{ay}^2)}{n_b^2-n_a^2}\left(\frac{\chi_{zy}}{\chi_{zz}}+2u_{ay}\right)^2+\frac{n_a^2s_{ax}^2}{\chi_{zz}}\right]\partial_X^2E_0-\left[\frac{n_a^2(s_{ax}^2+s_{ay}^2)}{n_b^2-n_a^2}\left(\frac{\chi_{zx}}{\chi_{zz}}+2u_{ax}\right)^2\right.$$
$$\left.+\frac{n_a^2s_{ay}^2}{\chi_{zz}}\right]\partial_Y^2E_0+2\left[\frac{n_a^2(s_{ax}^2+s_{ay}^2)}{n_b^2-n_a^2}\left(\frac{\chi_{zx}}{\chi_{zz}}+2u_{ax}\right)\left(\frac{\chi_{zy}}{\chi_{zz}}+u_{ay}\right)\right.$$
$$\left.-\frac{n_a^2s_{ax}s_{ay}}{\chi_{zz}}\right]\partial_X\partial_YE_0+2i\frac{k_a}{v_a}\partial_T(s_{ax}E_{1x}+s_{ay}E_{1y})-k_a\left(\frac{\chi_{zx}}{\chi_{zz}}s'_{ax}+\frac{\chi_{zy}}{\chi_{zz}}s'_{ay}+s'_{az}\right)$$
$$\times(s_{ax}\partial_T\partial_XE_0+s_{ay}\partial_T\partial_YE_0). \tag{36}$$

By collecting the expressions (33)–(36) of the four terms which are coming into the identity (31), we conclude that Eq. (26) holds true. □

C. Anomalous diffraction for biradial waves in biaxial crystals

We examine in this section the propagation in biaxial crystals in the particular configuration $\theta=\theta_r(\omega)$ and $\phi=0$ or π where

$$\sin^2\theta_r=\frac{1-\chi_2/\chi_1}{1-\chi_3/\chi_1},$$

where we have assumed that $\chi_1>\chi_2>\chi_3$. Computing all relevant quantities we have found that the eigenindices are $n_a^2=\chi_2$ and $n_b^2=\chi_1\chi_3/(\chi_1+\chi_3-\chi_2)$. The corresponding unit polarization vectors are

$$\mathbf{s}_a=(\cos\beta_r,0,\sin\beta_r)^T,$$

where the angle β_r is given by

$$\tan\beta_r=-\frac{\sqrt{(\chi_2-\chi_3)(\chi_1-\chi_2)}}{\chi_1+\chi_3-\chi_2}.$$

The diffraction coefficients are:

$$c_{a,xx}=1,\quad c_{a,xy}=0,\quad c_{a,yy}=0,$$

$$c_{b,xx}=\frac{\chi_1\chi_3}{(\chi_1+\chi_3-\chi_2)^2},\quad c_{b,xy}=0,\quad c_{b,yy}=\frac{\chi_1+\chi_3}{\chi_1+\chi_3-\chi_2}.$$

The striking point is that the diffraction operator for the type a wave is degenerate, in the sense that there is no diffraction in the y direction. The a wave is polarized along the y axis and satisfies the Schrödinger equation

$$2ik_a\partial_\zeta E_{0y}+\partial_X^2 E_{0y}-\sigma_a\partial_T^2 E_{0y}=0.$$

Such an anomalous behavior is consistent with the results derived in Ref. 11 where the authors show that the asymptotic form of the Green function is proportional to $z^{-1/2}$ instead of the standard z^{-1}-decay. This phenomenon is made transparent from our results, since the solution of a Schrödinger equation with a d-dimensional second-order operator spreads out as $z^{-d/2}$. This configuration could involve an interesting application in nonlinear optics. Consider high-intensity pulses so that the nonlinearity of the medium should be taken into account. Choose a carrier frequency such that the phase matching condition for the second-harmonic generation is not fulfilled. We may then expect that the main nonlinear term reads as a cubic Kerr effect, so that the field should satisfy (neglecting group velocity dispersion):

$$2ik_a\partial_\zeta E_{0y}+\partial_X^2 E_{0y}+\gamma|E_{0y}|^2 E_{0y}=0.$$

This one-dimensional nonlinear Schrödinger equation possesses the complete integrability property, which implies that stable solitons should be generated and propagate over large distances. This configuration will be studied in the companion paper.[7]

VI. UNIAXIAL CRYSTALS

We assume in this section that the dielectric tensor $\chi^{(1)}$ corresponds to the uniaxial case, that is to say $\chi_1=\chi_2:=\chi_o$ and $\chi_3=\chi_e$, with $\chi_e\neq\chi_o$. The results derived in the above sections can then be rewritten in simpler terms. In the general framework $\theta\neq 0$ there are two distinct eigenindices, the so-called ordinary and extraordinary refractive indices

$$n_o(\omega)=\chi_o(\omega)^{1/2},\quad n_e(\omega)=\left(\frac{\chi_o(\omega)\chi_e(\omega)}{\cos^2\theta\chi_e(\omega)+\sin^2\theta\chi_o(\omega)}\right)^{1/2}. \tag{37}$$

These configurations correspond to an ordinary wave and an extraordinary wave, respectively. The unit polarization vector of an ordinary wave is simply $\mathbf{s}_o=(0,1,0)^T$, while the polarization vector of an extraordinary wave lies in the plane (xz) and is given by $\mathbf{s}_e(\omega)=(\cos\beta,0,\sin\beta)^T$, where the angle $\beta(\omega)$ is

$$\tan\beta(\omega)=\frac{\cos\theta\sin\theta(\chi_e(\omega)-\chi_o(\omega))}{\cos^2\theta\chi_e(\omega)+\sin^2\theta\chi_o(\omega)}.$$

A. Ordinary wave

The vector $\mathbf{u}_o$ which gives the direction of the rays along which the wave propagates in the geometric framework is simply $\mathbf{u}_o=(0,0,-1)^T$. The diffraction coefficients are $c_{o,xy}=0$, $c_{o,xx}=c_{o,yy}=1$, and $\mathbf{u}_o'=\mathbf{0}$, so that the propagation equation in the moving frame $(\delta(t-z/v_o),\delta x,\delta y,\delta^2 z))$ reads as the standard Schrödinger equation:

$$2ik_o\partial_\zeta E_{0y}+\partial_X^2 E_{0y}+\partial_Y^2 E_{0y}-\sigma_o\partial_T^2 E_{0y}=0.$$

An ordinary wave propagates according to the usual rules which govern the propagation of waves in linear isotropic media.

B. Extraordinary wave

The Poynting vector is $\mathbf{u}_e = (\tan\beta, 0, -1)^T$. Thus β is the walk-off angle, that is to say the angle between the carrier wave vector $\mathbf{k}$ and the Poynting vector. The transverse diffraction coefficient $c_{e,xy}$ is zero while

$$c_{e,xx}(\omega) = \frac{\chi_o \chi_e}{(\cos^2\theta\chi_e + \sin^2\theta\chi_o)^2}, \quad c_{e,yy}(\omega) = \frac{\chi_o}{\cos^2\theta\chi_e + \sin^2\theta\chi_o}.$$

Furthermore $u'_{e,x} = (\tan\beta)'$ and $u'_{e,y} = 0$. The extraordinary wave $E_0 = \mathbf{s}_e . \mathbf{E}_0$ in the moving frame $(\delta(t - z/v_e), \delta(x + \tan\beta z), \delta y, \delta^2 z))$ thus satisfies the equation

$$2ik_e \partial_\zeta E_0 + c_{e,xx}\partial_X^2 E_0 + c_{e,yy}\partial_Y^2 E_0 + 2k_e(\tan\beta)'\partial_T\partial_X E_0 - \sigma_e \partial_T^2 E_0 = 0.$$

In a negative crystal $\chi_o > \chi_e$ we have $c_{e,xx} < c_{e,yy}$, which proves that diffraction effects are more important along the y axis than along the x axis. Note that, as $\theta \to 0$, we have $n_e \to n_o$ while the coefficients $c_{e,xx}$ and $c_{e,yy}$ converge to χ_o/χ_e. But as pointed out in Sec. V B, this extrapolation is not correct since terms of order δ are inversely proportional to $n_o - n_e$ and consequently tend to infinity. At the limit $\theta \to 0$ new terms of order 1 must be introduced. This will be done in the next section. Finally note that the expressions of the diffraction coefficients $c_{m,\ldots}$, $m = o, e$, are compatible with the asymptotic form of the Green function in the case of a uniaxial medium given by Lax and Nelson.[12]

VII. CRITICAL CONFIGURATION IN UNIAXIAL CRYSTALS

As demonstrated in Sec. IV C, if $\chi_1 = \chi_2 := \chi_o$ and $\chi_3 := \chi_e$ there exists a family of critical configurations characterized by $\theta = 0$ and any ϕ for which the indices n_a and n_b are both equal to $\sqrt{\chi_o}$. We assume in this section that the crystal is uniaxial and tailored so that its principal axis and the propagation axis z of the input pulse are collinear. In such a configuration there is no distinction between ordinary and extraordinary waves, and the field is transverse (there is no component E_{0z}). Further the propagations of the components E_{0x} and E_{0y} which used to be independent in the configuration $\theta \neq 0$ are now coupled since they have the same rapid phase $(\omega, n_o\omega/c)$. To deal with this coupling we consider a general form for the source

$$\mathcal{S} = \tfrac{1}{2}(v_x(\delta x, \delta y, \delta t), v_y(\delta x, \delta y, \delta t), 0)^T e^{-i\omega t} + cc. \tag{38}$$

The dispersion relationship, group velocity coefficient and dispersion are similar to those of a standard ordinary wave and given by (37).

A. Geometric optics

We adopt the ansatz (7) with $H = \{(\omega, k_o(\omega))\}$

$$\mathcal{E} = \frac{1}{2}\left(\sum_{j=0}^{\infty} \delta^j \mathbf{E}_j(\delta t, \delta x, \delta y, \delta z)\right) e^{i(k_o(\omega)z - \omega t)} + cc.$$

We denote $T = \delta t$, $X = \delta x$, $Y = \delta y$, $Z = \delta z$, and $k_o = k_o(\omega)$.

Proposition 3: If the source $\mathcal{S}$ can be expanded as (38), then the leading order term $\mathbf{E}_0$ of the slowly varying envelope is transverse. It has the same polarization as the incoming pulse, and it satisfies the transport equation

$$\partial_Z \mathbf{E}_0 + v_o^{-1}\partial_T \mathbf{E}_0 = 0, \tag{39}$$

starting from $\mathbf{E}_0(T,X,Y,Z=0)=(2/[1+n_o(\omega)])\mathbf{v}(T,X,Y)$.

In the geometric optics framework, this proposition demonstrates that the leading order terms of the components of the wave with perpendicular polarizations propagate without interaction according to the standard law of ordinary waves.

B. Diffractive optics

Equation (39) describe the propagation of the slowly varying envelope of the wave over propagation distance z of the order of δ^{-1}. It appears that the envelope is transported without deformation. Accordingly, if one considers the envelope in the moving reference frame $(\delta(t-z/v_o),\delta x,\delta y)$, then no evolution is noticeable as far as $z\sim\delta^{-1}$. It is therefore, necessary to address the problem of long-range propagation, over distances z of the order of δ^{-2}. The corresponding ansatz is the following:

$$\mathcal{E}=\frac{1}{2}\left(\sum_{j=0}^{\infty}\delta^j\mathbf{E}_j(\delta(t-z/v_o),\delta x,\delta y,\delta^2 z)\right)e^{i(k_o(\omega)z-\omega t)}+cc. \tag{40}$$

We denote $T=\delta(t-z/v_o)$, $X=\delta x$, $Y=\delta y$ and the long-range scale is represented by the slow variable $\zeta=\delta^2 z$.

Proposition 4: If the source $\mathcal{S}$ can be expanded as (38), then the leading order term $\mathbf{E}_0$ of the slowly varying envelope is elliptically polarized in the plane (x, y) and it obeys the coupled system of Schrödinger equations in the moving frame:

$$2ik_o\partial_\zeta E_{0x}+\rho\partial_X^2E_{0x}+\partial_Y^2E_{0x}+(\rho-1)\partial_X\partial_YE_{0y}-\sigma_o\partial_T^2E_{0x}=0, \tag{41a}$$

$$2ik_o\partial_\zeta E_{0y}+\partial_X^2E_{0y}+\rho\partial_Y^2E_{0y}+(\rho-1)\partial_X\partial_YE_{0x}-\sigma_o\partial_T^2E_{0y}=0, \tag{41b}$$

starting from $\mathbf{E}_0(T,X,Y,Z=0)=(2/[1+n_o(\omega)])\mathbf{v}(T,X,Y)$, *where* $\rho(\omega):=\chi_o(\omega)/\chi_e(\omega)$.

Note that the result of this proposition was reported in Ref. 13, which is as far as we know the only paper which provides an explicit form for the diffraction operator in an anisotropic medium. Nevertheless Ref. 13 only addressed the propagation of pulses along the principal axis of a uniaxial crystal, while our formulas are valid for more general configurations and systems (41a) and (41b) is just a particular application. As in the diffractive regime of the propagation of an extraordinary wave in the framework $\theta\neq 0$, we find an anisotropic diffraction operator. Further systems (41a) and (41b) puts into evidence a coupling between the linear Schrödinger type equations satisfied by the components of the field which are polarized along the x and y axes, respectively. This coupling shows itself in crossed second-order spatial derivatives which act onto the orthogonally polarized components. Accordingly, if the input wave is linearly polarized, then the spatial spectrum of the orthogonally polarized output field will present a dark cross which is known as the Maltese cross.

C. Almost-critical configuration $\theta=\delta\eta$

The condition $\theta=0$ is very stringent. It seems hardly possible to reach such a perfect level in realistic experimental configurations. It is consequently relevant to address the problem of the influence of a small perturbation of the ideal case $\theta=0$ by considering that the main optic axis of the crystal is collinear to the propagation axis up to a term of order δ. The second motivation of this section is to make smooth the transition between the results of the cases $\theta\neq 0$ and $\theta=0$, since it appears at first glance that there is discontinuity. As we shall see in this section, this apparent disagreement is involved by the fact that the transition is continuous at $\theta=0$ when considering a change of θ at rate δ. Accordingly we set $\theta=\delta\eta$. In the geometric optics framework, one finds the very same equations as in the case $\eta=0$. In the diffractive optics framework, one finds the following perturbed Schrödinger equations:

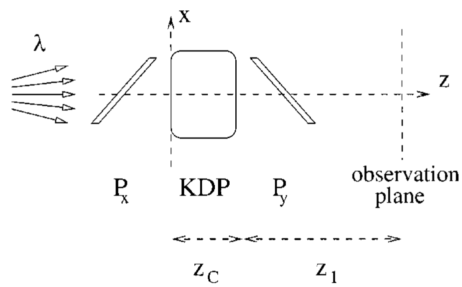

FIG. 2. Experimental setup of a Pockels cell.

$$2ik_o\partial_\zeta E_{0x}+\rho\partial_X^2E_{0x}+\partial_Y^2E_{0x}+(\rho-1)\partial_X\partial_Y E_{0y}-\sigma_o\partial_T^2E_{0x}$$
$$=\eta^2k_o^2(\rho-1)E_{0x}-i\eta k_o(\rho-1)(2\partial_X E_{0x}+\partial_Y E_{0y}), \qquad (42a)$$

$$2ik_o\partial_\zeta E_{0y}+\partial_X^2E_{0y}+\rho\partial_Y^2E_{0y}+(\rho-1)\partial_X\partial_Y E_{0x}-\sigma_o\partial_T^2E_{0y}=-i\eta k_o(\rho-1)\partial_Y E_{0x}. \qquad (42b)$$

These equations involve an interesting application that we discuss in the next section.

D. Application: Detection of the optic axis of a crystalline medium

This section is devoted to a useful and straightforward application of the propagation equations derived here above. We aim at determining the optic axis of a uniaxial crystal by a simple and efficient method. The technique which is described here below is widely used to bring into alignment Pockels cells in experimental setups. We consider the experimental configuration presented in Fig. 2. A linearly polarized divergent light beam emerging from a polarizer P_x is normally incident onto a plane parallel crystal plate of thickness z_c. The optic axis of the crystal is assumed to be almost collinear to the propagation axis z, and we are looking for the angle mismatch η between these axes.

We consider in this section long pulses with carrier wavelength λ so that the time-dependence of the envelope is much slower than its transverse spatial dependence and can be neglected. We take the Fourier transform with respect to the transverse spatial coordinates

$$\hat{\mathbf{E}}_0=\int \mathbf{E}_0 e^{-i(k_x x+k_y y)}dxdy.$$

Inside the crystal plate the field evolution is ruled by the systems (42a) and (42b) which reduces to a system of ordinary differential equations (we drop the 0-index)

$$2ik_o\partial_z\hat{E}_x=\rho k_x^2\hat{E}_x+k_y^2\hat{E}_x+(\rho-1)k_xk_y\hat{E}_y+(\rho-1)\eta k_o((\eta k_o-2k_x)\hat{E}_x-k_y\hat{E}_y),$$

$$2ik_o\partial_z\hat{E}_y=k_x^2\hat{E}_y+\rho k_y^2\hat{E}_y+(\rho-1)k_xk_y\hat{E}_x-(\rho-1)\eta k_o k_y\hat{E}_x,$$

that can be solved exactly by a straightforward exponentiation. The input field is linearly polarized along the x axis by the P_x polarizer, and the P_y polarizer eliminates the x component of the output field. Consequently the spectral intensity of the output field is

$$|\hat{E}_y|^2(z_c,k_x,k_y)=F\left((k_x-\eta k_o)\sqrt{\frac{(\rho-1)z_c}{4k_o}},\quad k_y\sqrt{\frac{(\rho-1)z_c}{4k_o}}\right)|\hat{E}_x|^2(0,k_x,k_y), \qquad (43a)$$

$$F(u,v)=4u^2v^2\,\mathrm{sinc}^2(u^2+v^2), \qquad (43b)$$

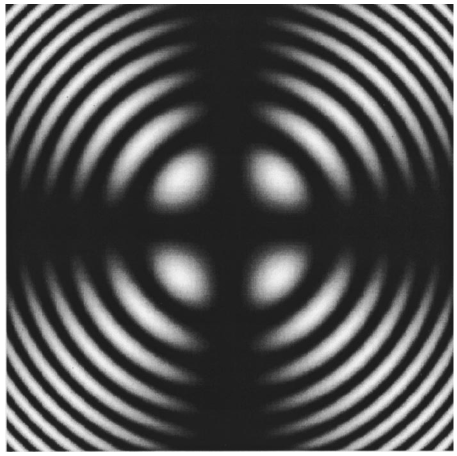

FIG. 3. Interference pattern from uniaxial crystal plate cut perpendicular to the optic axis, between two crossed polarizers. Function $(u,v)\mapsto F(u,v)$ over the domain $(-4,4)\times(-4,4)$.

where $\mathrm{sinc}(s)=\sin(s)/s$. The function F is plotted in Fig. 3. From z_c to $z_{\mathrm{obs}}:=z_c+z_1$ the pulse propagates in vacuum, so in the far field configuration $(z_1\gg\lambda)$ the intensity distribution $|E_y|^2(z_{\mathrm{obs}},x,y)$ is proportional to the power spectral density of the near field:

$$|E_y|^2(z_{\mathrm{obs}},x_o,y_o)=\frac{1}{\lambda^2 z_1^2}|\hat{E}_y|^2\left(z_c,\frac{2\pi x_o}{\lambda z_1},\frac{2\pi y_o}{\lambda z_1}\right).$$

Substituting Eq. (43a) into this identity yields

$$|E_y|^2(z_{\mathrm{obs}}x_o,y_o)=\frac{1}{\lambda^2 z_1^2}F\left(\left(\frac{x_o}{z_1}+\eta n_o\right)\sqrt{\frac{(\rho-1)\pi z_c}{2n_o\lambda}},\quad \frac{y_o}{z_1}\sqrt{\frac{(\rho-1)\pi z_c}{2n_o\lambda}}\right)|\hat{E}_x|^2(0,k_x,k_y).$$

Conclusion. If the optic axis of the crystal is perfectly collinear to the propagation axis of the beam, then the far field intensity presents a centered dark cross. If there exists an angle mismatch η between the optic axis and the propagation axis, then the cross is shifted by the quantity $\Delta x_o=\eta n_o z_1$.

VIII. CONICAL REFRACTION IN BIAXIAL CRYSTAL

As pointed out in Sec. IV C, if $\chi_1>\chi_2>\chi_3$ are distinct, then a simple study of the matrix χ shows that there exists only one case when $n_a=n_b$, which corresponds to the configuration when $\phi=0$ and $\theta=\pm\theta_c(\omega)$ where $\theta_c(\omega)$ is defined by (17). In such a configuration $n_a=n_b=\chi_2^{1/2}$, and the two mutually orthogonal polarization vectors are:

$$\mathbf{s}_a=(\cos\beta_c,0,\sin\beta_c)^T,\quad \mathbf{s}_b=(0,1,0)^T,$$

where the angle $\beta_c(\omega)$ between the polarization vector $\mathbf{s}_a$ and the propagation axis z is given by

$$\tan\beta_c=-\sqrt{\left(1-\frac{\chi_2}{\chi_1}\right)\left(\frac{\chi_2}{\chi_3}-1\right)}.$$

Substituting the ansatz (19) into Eq. (2) and collecting the coefficients of each power of δ, we get the following result.

Proposition 5: If we denote the projections of the field $\mathbf{E}_0$ *onto the vectors* $\mathbf{s}_a$ *and* $\mathbf{s}_b$ *by* $E_{0a}=\mathbf{s}_a.\mathbf{E}_0$ *and* $E_{0b}=\mathbf{s}_b.\mathbf{E}_0$, *then the scalar fields* E_{0a} *and* E_{0b} *satisfy the coupled equations*

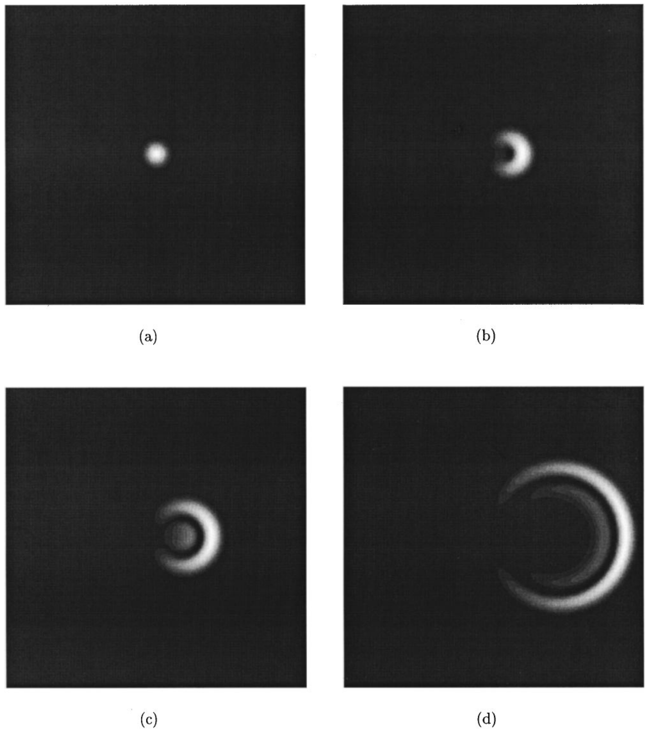

FIG. 4. Conical refraction of a Gaussian pulse polarized along the x axis for different values of Z. $Z=0$ (a), $Z=4.13$ (b), $Z=8.26$ (c), and $Z=20.65$ (d). Here $\cos\beta_c=0.9$ or equivalently $(\tan\beta_c)/2=0.24$.

$$v_a^{-1}\partial_T E_{0_a}+\partial_Z E_{0a}-\tan\beta_c\partial_X E_{0a}=\frac{\tan\beta_c}{2\cos\beta_c}\partial_Y E_{ob}, \tag{44a}$$

$$v_a^{-1}\partial_T E_{0b}+\partial_Z E_{0b}=\frac{\sin\beta_c}{2}\partial_Y E_{0a}, \tag{44b}$$

starting from $E_{0a}(T,X,Y,Z=0)=(2/[1+n_a(\omega)])v_x(T,X,Y)$ *and* $E_{0b}(T,X,Y,Z=0)=(2/[1+n_a(\omega)])v_y(T,X,Y)$.

Taking the spatial Fourier transform with respect to the transverse coordinates (X,Y), the solution field reads

$$\begin{pmatrix}\hat{E}_{0a}\\ \hat{E}_{0b}\end{pmatrix}(Z,T,k_x,k_y)=\frac{2e^{ik_xZ'}}{1+n_a}\left[\cos(k_rZ')\begin{pmatrix}1&0\\0&1\end{pmatrix}+i\sin(k_rZ')\frac{1}{k_r}\begin{pmatrix}k_x&\dfrac{k_y}{\cos\beta_c}\\ k_y\cos\beta_c&-k_x\end{pmatrix}\right]\hat{\mathbf{v}}(T-Z/v_a,k_x,k_y),$$

where $k_r=\sqrt{k_x^2+k_y^2}$ and $Z'=Z(\tan\beta_c)/2$. The evolution of an input Gaussian field is plotted in Fig. 4. Let us study the field in the framework $|Z'|>1$ but not so large so as to be allowed to neglect the diffractive terms (i.e., $\delta^{-1}\leqslant z\ll\delta^{-2}$). In the physical space, using the stationary phase method, we get that the field is concentrated on the circle with center Z' and radius Z' if the input field is localized around 0. In other words the wave surface has the shape of a cone. This conical refraction is a well-known phenomena which was predicted in 1832 by Hamilton and observed thereafter by Lloyd. Historical references and an elementary study of conical refraction can be found in Ref. 1. More advanced treatments are devoted to the subject.[14,15] In particular Warnick and Arnold[16] predicts additional fringes by computing the asymptotic form of the Green function. These results can be exhibited in our framework quite easily. Let us denote by (cos γ,sin γ,0) the unit polarization vector of the source **v**, and by R_0 the radius of the input beam. If (X,Y) is farther than R_0 from the cone $(X-Z')^2+Y^2=Z'^2$, then we have $|E_0|^2(Z,X,Y)=o(Z^{-1})$. If (X,Y) is close to the cone $(X-Z')^2+Y^2=Z'^2$ by less than R_0, then denoting $X=Z'+Z'\cos\alpha$ and $Y=Z'\sin\alpha$

$$|E_0|^2(Z,X,Y)\simeq\frac{Z_0}{Z}\left(\cos\left(\frac{\alpha}{2}\right)\cos(\gamma)\cos(\beta_c)+\sin\left(\frac{\alpha}{2}\right)\sin(\gamma)\right)^2\left(1+\cos^2\left(\frac{\alpha}{2}\right)\tan^2(\beta_c)\right),$$

where Z_0 is a characteristic distance proportional to $R_0/\tan\beta_c$. If Z is so large that it reaches values of order δ^{-1} (i.e., z reaches values of order δ^{-2}), then one should take into account the second-order derivatives, which makes the evolution of the field more complicated. The strategy is still the same as in the other configurations. It consists in looking at the evolution of the field at the long scale $\delta^2 z$ around the points defined by the transport equation, that is to say the cone defined by $(x-z(\tan\beta_c)/2)^2+y^2=(z(\tan\beta_c)/2)^2$. This specific study will be carried out elsewhere. Nevertheless, we would like to add the following comment that gives a new insight into the phenomena that govern conical refraction. In the moving reference frame $(\delta(t-z/v_a),\delta(x+z(\tan\beta_c)/2),\delta y,\delta z)$ Eqs. (44a) and (44b) read as

$$\partial_Z E_{0a}=\frac{\tan\beta_c}{2}\partial_X E_{0a}+\frac{\tan\beta_c}{2\cos\beta_c}\partial_Y E_{0b}, \tag{45a}$$

$$\partial_Z E_{0b}=\frac{\sin\beta_c}{2}\partial_Y E_{0a}-\frac{\tan\beta_c}{2}\partial_X E_{0b}. \tag{45b}$$

Composing these equations establishes that the modes E_{0m} for $m=a$ and b obey the standard wave equations with uniform ''velocity'' $(\tan\beta_c)/2$

$$\partial_Z^2 E_{0a}=\frac{\tan^2\beta_c}{4}(\partial_X^2+\partial_Y^2)E_{0_a}, \tag{46a}$$

$$\partial_Z^2 E_{0b}=\frac{\tan^2\beta_c}{4}(\partial_X^2+\partial_Y^2)E_{0b}, \tag{46b}$$

where Z plays the role of the usual time. The initial conditions are imposed by $E_{0m}(Z=0)$ and $\partial_Z E_{0m}(Z=0)$. As is well-known the solution of the wave equation $u_{tt}=c^2\Delta u$ satisfies the Huygens principle which states that, if the Laplacian acts on a space with odd dimension d, then the solution $u(x,t)$ depends only on the initial data at $t=0$ for $x_0\in\{x_0\in\mathbb{R}^d,|x_0-x|=ct\}$. Thus an initial delta-like pulse at $t=0$, $x=0$ will give rise at time t to a pulse concentrated on the circle with center 0 and radius ct. This property does not hold true for even dimension, since the solution $u(x,t)$ then depends on the initial data at $t=0$ in the cone $x_0\in\{x_0\in\mathbb{R}^d,|x_0-x|\leq t\}$. In the standard wave equation, the space has dimension $d=3$ and the Huygens principle is satisfied. In our case, Z plays the role of t and $d=2$, which proves that complex structure inside the main cone can be generated during the propagation. We refer to the standard literature on the wave equation for a description of the different phenomena that can arise.[17,18]

IX. TRANSITION BETWEEN UNIAXIAL AND BIAXIAL CONFIGURATIONS

We assume in this section that the crystal is tailored so that its principal axis and the propagation axis z of the input pulse are collinear. Furthermore the susceptibilities χ_1 and χ_2 are close to each other so that they can be written as: $\chi_1=\chi_o$ and $\chi_2=\chi_o-\delta^2\eta_\chi$, where η_χ is of order 1. Accordingly the tensor χ in the (x, y, z)-frame reads as

$$\chi=\chi^0-\delta^2\eta_\chi M_\phi, \quad \chi^0:=\begin{pmatrix}\chi_o & 0 & 0\\ 0 & \chi_o & 0\\ 0 & 0 & \chi_e\end{pmatrix}, \quad M_\phi=N_\phi\oplus 0, \quad N_\phi:=\begin{pmatrix}\sin^2\phi & \cos\phi\sin\phi\\ \cos\phi\sin\phi & \cos^2\phi\end{pmatrix}.$$

In such a configuration the dispersion equation is the same as in the uniaxial case considered in Sec. VII since the mismatch only appears at order δ^2. Thus the two eigenindices are equal to $\chi_o^{1/2}$, the group velocity coefficient and dispersion are similar to those of a standard ordinary wave and given by (37), and the leading order term $\mathbf{E}_0$ of the field is transverse. Further the propagations of the components E_{0x} and E_{0y} are coupled. To deal with this coupling we consider the general form (38) for the source. The weak biaxial property (of order δ^2) does not involve any modification of the transport equations which govern the propagation of the wave in the geometric scales with respect to the uniaxial case. We thus consider the scales of diffractive optics and we adopt the ansatz (40).

Proposition 6: If the source $\mathcal{S}$ can be expanded as (38), then the leading order term $\mathbf{E}_0$ of the slowly varying envelope is transverse and it satisfies the coupled Schrödinger equations:

$$2ik_o\partial_\zeta E_{0x}+\rho\partial_X^2E_{0x}+\partial_Y^2E_{0x}+(\rho-1)\partial_X\partial_YE_{0y}-\sigma_o\partial_T^2E_{0x}=k_o^2\eta_\chi(M_\phi\mathbf{E}_0)_x, \tag{47a}$$

$$2ik_o\partial_\zeta E_{0y}+\partial_X^2E_{0y}+\rho\partial_Y^2E_{0y}+(\rho-1)\partial_X\partial_YE_{0x}-\sigma_o\partial_T^2E_{0y}=k_o^2\eta_\chi(M_\phi\mathbf{E}_0)_y, \tag{47b}$$

starting from $\mathbf{E}_0(T,X,Y,\zeta=0)=(2/[1+n_o(\omega)])\mathbf{v}(T,X,Y)$, *where* $\rho=\chi_o/\chi_e$.

Taking the spatial Fourier transform with respect to the transverse coordinates (X, Y), the systems (47a) and (47b) reduces to a system of ordinary differential equations. If the initial pulse is polarized along the x axis, and if we retain only the component of the output pulse which is y polarized, then we have

$$|\hat{E}_y(z_c,k_x,k_y)|^2=F_\phi\left(k_x\sqrt{\frac{(\rho-1)z_c}{4k_o}},k_y\sqrt{\frac{(\rho-1)z_c}{4k_o}},\frac{\eta_\chi k_oz_c}{4}\right)|\hat{E}_x(0,k_x,k_y)|^2,$$

$$F_\phi(u,v,\eta)=(2uv+\eta\sin(2\phi))^2 \times\mathrm{sinc}^2\left(\sqrt{(u^2+v^2)^2+\eta^2+2\eta((v^2-u^2)\cos(2\phi)+2uv\sin(2\phi))}\right),$$

where $\mathrm{sinc}(s)=\sin(s)/s$. Figure 5 plots the function $(u,v)\mapsto F_\phi(u,v,\eta)$ for different values of the parameters ϕ and η. Comparisons with experimental observations show excellent agreement. See Figure 14.26 in Ref. 1 for an observation of Fig. 5(f), Figures 465 to 466 in Ref. 19 for observations of Figs. 5(d) and Fig. 5(e), and Fig. 5.30 in Ref. 20 for another observation of Fig. 5(e). In particular Fig. 5(f) is the theoretical counterpart of the cover of the sixth edition of the book ''Principles of Optics'' by Born and Wolf![1] Let us briefly discuss the main properties of the functions F_ϕ. If the initial polarization vector is collinear to one of the axis of the crystal ($\phi=0$ or π), then F_0 has a factor u^2v^2, which shows that there is a centered dark cross, whatever η. If the initial polarization vector is collinear to the bisecting line of the axes of the crystal ($\phi=\pi/4$), then we have

$$F_{\pi/4}(u,v,\eta)=(2uv+\eta)^2\mathrm{sinc}^2\left(\sqrt{(u^2+v^2)^2+\eta^2+4\eta uv}\right), \tag{48}$$

which shows that the transmission at the center gets nonzero when η increases and may even be 1 at some particular values (see Fig. 5). Indeed, whatever ϕ, the transmission at $u=v=0$ is $F_\phi(0,0,\eta)=\sin(\eta)^2\sin(2\phi)^2$. If $\phi=0$, it is always 0, but if $\phi=\pi/4$, it is equal to $\sin(\eta)^2$ which is maximal and equal to 1 when $\eta=\pi/2 \bmod \pi$. This implies that a plane wave is fully transmitted in this configuration.

The results derived in this section provide the principle and the precise characterization of electro-optic switching devices of the family of Pockels cells.[2] Indeed, the experimental setup

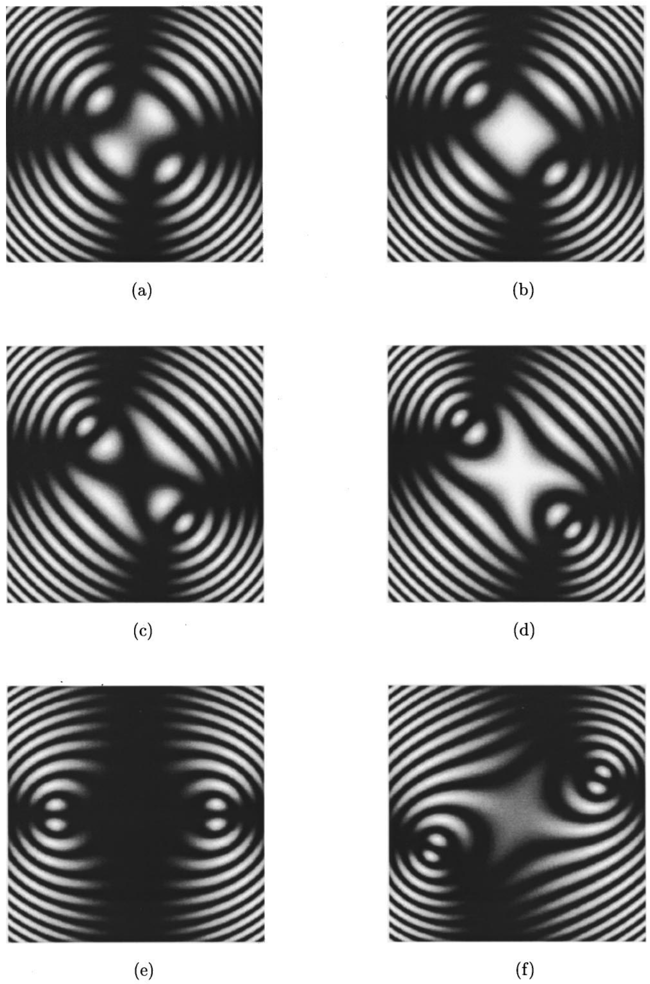

FIG. 5. Interference patterns from biaxial crystal plates between two crossed polarizers. Functions $(u,v) \mapsto F_{\phi}(\eta,u,v)$ over the domain $(-4,4)\times(-4,4)$ for different values of η and ϕ. $\eta=\pi/4$ and $\phi=\pi/4$ (a), $\eta=\pi/2$ and $\phi=\pi/4$ (b), $\eta=\pi$ and $\phi=\pi/4$ (c), $\eta=3\pi/2$ and $\phi=\pi/4$ (d), $\eta=2\pi$ and $\phi=\pi$ (e), and $\eta=5\pi/2$ and $\phi=7\pi/8$ (f). The case $\eta=0$ (and any ϕ) is plotted in Fig. 3.

depicted in Fig. 2 corresponds to the case $\eta_\chi=0$. Applying an electric field between two faces of the potassium dihydrogen phosphate (KDP) crystal plate involves an alteration of the distribution of the electric charges of the atoms and molecules which constitute the crystal, which affect the optical properties of the medium. The theory of electro-optics is well-known, and we refer for instance to Ref. 21, Section 87, for a survey. In the case of point group $\overline{4}2m$ to which KDP crystal belongs, it is known that the crystals become biaxial while they are uniaxial in the absence of external electric field, that is to say η_χ takes nonzero values which are imposed by the applied electric field. By applying the tension from 0 to the value corresponding to $\eta_\chi=2\pi/(k_o(\omega)z_c)$, the transmittivity goes from 0 to 1 for an input plane wave with carrier frequency ω. Finally note also that the transfer function $(u,v)\mapsto F_{\pi/4}(u,v,\pi/2)$ possesses a flat top hat. This configuration could then be used as a spatial filter as well.

X. CONCLUSION

In this paper we have derived the equations which govern the linear propagation of the slowly varying envelopes of pulses in a bulk medium presenting anisotropic properties. The strategy mainly consists in two steps. We first consider the Maxwell equations in the scales of optic geometric, that is to say for propagation distance of the same order as the radius of the beam or the duration of the pulse times the light velocity. In this framework the propagation equations read as transport equations, which actually give the propagation of the rays according to the law of geometric optics. Second we revisit the Maxwell equations in the moving frame indicated by the above-derived transport equations. In the scales of the geometric optics, the propagation equations are then trivial, which allows to consider larger propagation distances, of the order of the Rayleigh distance or the dispersion distance. In this framework the propagation equations read as Schrödinger-type equations, which actually give the propagations of the slowly varying envelope according to the law of diffractive optics.

By applying this methodology we have put into evidence that we can deal with many situations. We have recovered well-known results, but we have also exhibited closed form expressions for the diffraction operator which has led to original results regarding an anomalous diffraction in a very particular configuration. Another advantage of this method is that it can still be applied when we take into account the nonlinear susceptibility of the medium. This generalization is addressed in the companion paper.[7]

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